Advanced engineering mathematics kreyszig solution manual

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Correlation…551 Solution Manuals Of ADVANCED ENGINEERING MATHEMATICS By ERWIN KREYSZIG 9TH EDITION This is Downloaded From www.mechanical.tk Visit www.mechanical.tk For More Solution Manuals Hand Books And Much Much More INSTRUCTOR'S MANUAL FOR ADVANCED ENGINEERING MATHEMATICS imfm.qxd 9/15/05 12:06 PM Page i ISBN-13: 978-0-471-72647-0 ISBN-10: 0471-72647-8 Printed in the United States of America 10 9 8 7 6 5 4 3 2 1 Vice President and Publisher: Laurie Rosatone Editorial Assistant: Daniel Grace Associate Production Director: Lucille Buonocore Senior Production Editor: Ken Santor Media Editor: Stefanie Liebman Cover Designer: Madelyn Lesure Cover Photo: © John Sohm/Chromosohm/Photo Researchers This book was set in Times Roman by GGS Information Services and printed and bound by Hamilton Printing. The cover was printed by Hamilton Printing. This book is printed on acid free paper. imfm.qxd 9/15/05 12:06 PM Page iv PREFACE General Character and Purpose of the Instructor's Manual This Manual contains: (I) Detailed solutions of the even-numbered problems. (II) General comments on the purpose of each section and its classroom use, with mathematical and didactic information on teaching practice and pedagogical aspects. Some of the comments refer to whole chapters (and are indicated accordingly). Changes in Problem Sets The major changes in this edition of the text are listed and explained in the Preface of the book. They include global improvements produced by updating and streamlining chapters as well as many local improvements aimed at simplification of the whole text. Speedy orientation is helped by chapter summaries at the end of each chapter, as in the last edition, and by the subdivision of sections into subsections with unnumbered headings. Resulting effects of these changes on the problem sets are as follows. The problems have been changed. The large total number of more than 4000 problems has been retained, increasing their overall usefulness by the following: • Placing more emphasis on modeling and conceptual thinking and less emphasis on technicalities, to parallel recent and ongoing developments in calculus. • Balancing by extending problem sets that seemed too short and contracting others that were too long, adjusting the length to the relative importance of the material in a section, so that important issues are reflected sufficiently well not only in the text but also in the problems. Thus, the danger of overemphasizing minor techniques and ideas is avoided as much as possible. • Simplification by omitting a small number of very difficult problems that appeared in the previous edition, retaining the wide spectrum ranging from simple routine problems to more sophisticated engineering applications, and taking into account the "algorithmic thinking" that is developing along with computers. • Amalgamation of text, examples, and problems by including the large number of more than 600 worked-out examples in the text and by providing problems closely related to those examples. • Addition of TEAM PROJECTS, CAS PROJECTS, and WRITING PROJECTS, whose role is explained in the Preface of the book. • Addition of CAS EXPERIMENTS, that is, the use of the computer in "experimental mathematics" for experimentation, discovery, and research, which often produces unexpected results for open-ended problems, deeper insights, and relations among practical problems. These changes in the problem sets will help students in solving problems as well as in gaining a better understanding of practical aspects in the text. It will also enable instructors to explain ideas and methods in terms of examples supplementing and illustrating theoretical discussions—or even replacing some of them if so desired. imfm.qxd 9/15/05 12:06 PM Page v "Show the details of your work." This request repeatedly stated in the book applies to all the problem sets. Of course, it is intended to prevent the student from simply producing answers by a CAS instead of trying to understand the underlying mathematics. Orientation on Computers Comments on computer use are included in the Preface of the book. Software systems are listed in the book at the beginning of Chap. 19 on numeric analysis and at the beginning of Chap. 24 on probability theory. ERWIN KREYSZIG vi Instructor's Manual imfm.qxd 9/15/05 12:06 PM Page vi Comment on Order of Sections This section could equally well be presented later in Chap. 1, perhaps after one or two formal methods of solution have been studied. SOLUTIONS TO PROBLEM SET 1.2, page 11 2. Semi-ellipse x2/4 y2/9 13/9, y 0. To graph it, choose the y-interval large enough, at least 0 y 4. 4. Logistic equation (Verhulst equation; Sec. 1.5). Constant solutions y 0 and y 1_2. For these, y 0. Increasing solutions for 0 y(0) 1_2, decreasing for y(0) 1_2. 6. The solution (not of interest for doing the problem) is obtained by using dy/dx 1/(dx/dy) and solving dx/dy 1/(1 sin y) by integration, x c 2/(tan 1_2y 1); thus y 2 arctan ((x 2 c) /(x c)). 8. Linear ODE. The solution involves the error function. 12. By integration, y c 1/x. 16. The solution (not needed for doing the problem) of y 1/y can be obtained by separating variables and using the initial condition; y2/2 t c, y 2t 1. 18. The solution of this initial value problem involving the linear ODE y y t2 is y 4et t2 2t 2. 20. CAS Project. (a) Verify by substitution that the general solution is y 1 cex. Limit y 1 (y(x) 1 for all x), increasing for y(0) 1, decreasing for y(0) 1. (b) Verify by substitution that the general solution is x4 y4 c. More "square- shaped," isoclines y kx. Without the minus on the right you get "hyperbola-like" curves y4 x4 const as solutions (verify!). The direction fields should turn out in perfect shape. (c) The computer may be better if the isoclines are complicated; but the computer may give you nonsense even in simpler cases, for instance when y(x) becomes imaginary. Much will depend on the choice of x- and y-intervals, a method of trial and error. Isoclines may be preferable if the explicit form of the ODE contains roots on the right. SECTION 1.3. Separable ODEs. Modeling, page 12 Purpose. To familiarize the student with the first "big" method of solving ODEs, the separation of variables, and an extension of it, the reduction to separable form by a transformation of the ODE, namely, by introducing a new unknown function. The section includes standard applications that lead to separable ODEs, namely, 1. the ODE giving tan x as solution 2. the ODE of the exponential function, having various applications, such as in radiocarbon dating 3. a mixing problem for a single tank 4. Newton's law of cooling 5. Torricelli's law of outflow. Instructor's Manual 3 im01.qxd 9/21/05 10:17 AM Page 3 In reducing to separability we consider 6. the transformation u y/x, giving perhaps the most important reducible class of ODEs. Ince's classical book [A11] contains many further reductions as well as a systematic theory of reduction for certain classes of ODEs. Comment on Problem 5 From the implicit solution we can get two explicit solutions y c (6x2) representing semi-ellipses in the upper half-plane, and y c (6x2) representing semi-ellipses in the lower half-plane. [Similarly, we can get two explicit solutions x(y) representing semi-ellipses in the left and right half-planes, respectively.] On the x-axis, the tangents to the ellipses are vertical, so that y(x) does not exist. Similarly for x(y) on the y-axis. This also illustrates that it is natural to consider solutions of ODEs on open rather than on closed intervals. Comment on Separability An analytic function ƒ(x, y) in a domain D of the xy-plane can be factored in D, ƒ(x, y) g(x)h(y), if and only if in D, ƒxyƒ ƒxƒy [D. Scott, American Math. Monthly 92 (1985), 422–423]. Simple cases are easy to decide, but this may save time in cases of more complicated ODEs, some of which may perhaps be of practical interest. You may perhaps ask your students to derive such a criterion. Comments on Application Each of those examples can be modified in various ways, for example, by changing the application or by taking another form of the tank, so that each example characterizes a whole class of applications. The many ODEs in the problem set, much more than one would ordinarily be willing and have the time to consider, should serve to convince the student of the practical importance of ODEs; so these are ODEs to choose from, depending on the students' interest and background. Comment on Footnote 3 Newton conceived his method of fluxions (calculus) in 1665–1666, at the age of 22. Philosophiae Naturalis Principia Mathematica was his most influential work. Leibniz invented calculus independently in 1675 and introduced notations that were essential to the rapid development in this field. His first publication on differential calculus appeared in 1684. SOLUTIONS TO PROBLEM SET 1.3, page 18 2. dy/y2 (x 2)dx. The variables are now separated. Integration on both sides gives 1_2x 2 2x c*. Hence y . 2 x2 4x c 1 y 4 Instructor's Manual im01.qxd 9/21/05 10:17 AM Page 4 4. Set y 9x v. Then y v 9x. By substitution into the given ODE you obtain y v 9 v2. By separation, dx. Integration gives arctan x c*, arctan 3x c and from this and substitution of y v 9x, y 3 tan (3x c), y 3 tan (3x c) 9x. 6. Set u y/x. Then y xu, y u xu. Substitution into the ODE and subtraction of u on both sides gives y u xu, u xu . Separation of variables and replacement of u with y/x yields 2u du dx, u2 8 ln x c, y2 x2 (8 ln x c). 8. Put u y/x. Then y xu, y u xu. Substitute u into the ODE, drop xu on both sides, and divide by x2 to get xy xu x2u 1_2x 2u2 xu, u 1_2u 2. Separate variables, integrate, and solve algebraically for u: 1_2 dx, 1_ 2(x c*), u . Hence y xu . 10. By separation, y dy 4x dx. By integration, y2 4x2 c. The initial condition y(0) 3, applied to the last equation, gives 9 0 c. Hence y2 4x2 9. 12. Set u y/x. Then y u xu. Divide the given ODE by x2 and substitute u and u into the resulting equation. This gives 2u(u xu) 3u2 1. Subtract 2u2 on both sides and separate the variables. This gives 2xuu u2 1, . Integrate, take exponents, and then take the square root: ln (u2 1) ln x c*, u2 1 cx, u cx 1. Hence y xu xcx 1. From this and the initial condition, y(1) c 1 2, c 5. This gives the answer y x5x 1. dx x du u2 1 2x c x 2 c 1 u du u2 8 x 4 u 4 u y x 4x y v 3 v 3 1 3 dv v2 9 Instructor's Manual 5 im01.qxd 9/21/05 10:17 AM Page 5 36. B now depends on h, namely, by the Pythagorean theorem, B(h) r2 (R2 (R h)2) (2Rh h2). Hence you can use the ODE h 26.56(A/B)h in the text, with constant A as before and the new B. The latter makes the further calculations different from those in Example 5. From the given outlet size A 5 cm2 and B(h) we obtain 26.56 h. Now 26.56 5/ 42.27, so that separation of variables gives (2Rh1/2 h3/2) dh 42.27 dt. By integration, 4_ 3Rh 3/2 2_5h 5/2 42.27t c. From this and the initial condition h(0) R we obtain 4_ 3R 5/2 2_5R 5/2 0.9333R5/2 c. Hence the particular solution (in implicit form) is 4_ 3Rh 3/2 2_5h 5/2 42.27t 0.9333R5/2. The tank is empty (h 0) for t such that 0 42.27t 0.9333R5/2; hence t 0.0221R5/2. For R 1 m 100 cm this gives t 0.0221 1005/2 2210 [sec] 37 [min]. The tank has water level R/2 for t in the particular solution such that R 0.9333R5/2 42.27t. The left side equals 0.4007R5/2. This gives t R5/2 0.01260R5/2. For R 100 this yields t 1260 sec 21 min. This is slightly more than half the time needed to empty the tank. This seems physically reasonable because if the water level is R/2, this means that 11/16 of the total water volume has flown out, and 5/16 is left—take into account that the velocity decreases monotone according to Torricelli's law. Problem Set 1.3. Tank in Problem 36 R R r h h 0.4007 0.9333 42.27 R5/2 25/2 2 5 R3/2 23/2 4 3 0.9333 42.27 5 (2Rh h2) dh dt 8 Instructor's Manual im01.qxd 9/21/05 10:17 AM Page 8 SECTION 1.4. Exact ODEs. Integrating Factors, page 19 Purpose. This is the second "big" method in this chapter, after separation of variables, and also applies to equations that are not separable. The criterion (5) is basic. Simpler cases are solved by inspection, more involved cases by integration, as explained in the text. Comment on Condition (5) Condition (5) is equivalent to (6) in Sec. 10.2, which is equivalent to (6) in the case of two variables x, y. Simple connectedness of D follows from our assumptions in Sec. 1.4. Hence the differential form is exact by Theorem 3, Sec. 10.2, part (b) and part (a), in that order. Method of Integrating Factors This greatly increases the usefulness of solving exact equations. It is important in itself as well as in connection with linear ODEs in the next section. Problem Set 1.4 will help the student gain skill needed in finding integrating factors. Although the method has somewhat the flavor of tricks, Theorems 1 and 2 show that at least in some cases one can proceed systematically—and one of them is precisely the case needed in the next section for linear ODEs. SOLUTIONS TO PROBLEM SET 1.4, page 25 2. (x y) dx (y x) dy 0. Exact; the test gives 1 on both sides. Integrate x y over x: u 1_2x 2 xy k(y). Differentiate this with respect to y and compare with N: uy x k y x. Thus k y, k 1_2 y 2 c. Answer: 1_2x 2 xy 1_2 y 2 1_2(x y) 2 c; thus y x c. 4. Exact; the test gives ey ex on both sides. Integrate M with respect to x to get u xey yex k(y). Differentiate this with respect to y and equate the result to N: uy xe y ex k N xey ex. Hence k 0, k const. Answer: xey yex c. 6. Exact; the test gives ex sin y on both sides. Integrate M with respect to x: u ex cos y k(y). Differentiate: uy ex sin y k. Equate this to N ex sin y. Hence k 0, k const. Answer: ex cos y c. 8. Exact; 1/x2 1/y2 on both sides of the equation. Integrate M with respect to x: u xy2 k(y). Differentiate this with respect to y and equate the result to N: uy K N, k 2y, k y 2. Answer: x2 y2 c. y x x y x y 1 x x y 2 y x y y x Instructor's Manual 9 im01.qxd 9/21/05 10:17 AM Page 9 10. Exact; the test gives 2x sin (x2) on both sides. Integrate N with respect to y to get u y cos (x2) l(x). Differentiate this with respect to x and equate the result to M: ux 2xy sin (x 2) l M 2xy sin (x2), l 0. Answer: y cos (x2) c. 12. Not exact. Try Theorem 1. In R you have Py Qx e xy 1 exy(x 1) xexy 1 Q so that R 1, F ex, and the exact ODE is (ey yex) dx (xey ex) dy 0. The test gives ey ex on both sides of the equation. Integration of M FP with respect to x gives u xey yex k(y). Differentiate this with respect to y and equate it to the result to N: uy xe y ex k N xey ex. Hence k 0. Answer: xey yex c. 14. Not exact; 2y y. Try Theorem 1; namely, R (Py Qx) /Q (2y y) /(xy) 3/x. Hence F 1/x 3. The exact ODE is (x) dx dy 0. The test gives 2y/x3 on both sides of the equation. Obtain u by integrating N FQ with respect to y: u l(x). Thus ux l M x . Hence l x, l x2/2, y2/2x2 x2/2 c*. Multiply by 2 and use the initial condition y(2) 1: x2 c 3.75 because inserting y(2) 1 into the last equation gives 4 0.25 3.75. 16. The given ODE is exact and can be written as d(cos xy) 0; hence cos xy c, or you can solve it for y by the usual procedure. y(1) gives 1 c. Answer: cos xy 1. 18. Try Theorem 2. You have R* (Qx Py) /P [cos xy x sin xy (x sin xy)]P . Hence F* y. This gives the exact ODE (y cos xy x) dx (y x cos xy) dy 0. 1 y x y2 1 y y2 x2 y2 x3 y2 x3 y2 2x2 y x2 y2 x3 10 Instructor's Manual im01.qxd 9/21/05 10:17 AM Page 10 SOLUTIONS TO PROBLEM SET 1.5, page 32 4. The standard form (1) is y 4y x, so that (4) gives y e4x [e4xx dx c] ce4x x/4 1/16. 6. The standard form (1) is y y . From this and (4) we obtain, with c 2 from the initial condition, y x3 [x3x3 dx c] x3[x c] x2 2x3. 8. From (4) with p 2, h 2x, r 4 cos 2x we obtain y e2x [e2x 4 cos 2x dx c] e2x[e2x(cos 2x sin 2x) c]. It is perhaps worthwhile mentioning that integrals of this type can more easily be evaluated by undetermined coefficients. Also, the student should verify the result by differentiation, even if it was obtained by a CAS. From the initial condition we obtain y(1_4) ce /2 0 1 2; hence c e /2. The answer can be written y e/22x cos 2x sin 2x. 10. In (4) we have p 4x2; hence h 4x3/3, so that (4) gives y e4x 3/3 [e(4x 3/3)x2 2/ 2 (4x2 x) dx c]. The integral can be evaluated by noting that the factor of the exponential function under the integral sign is the derivative of the exponent of that function. We thus obtain y e4x 3/3 [e(4x 3/3)x2 2/ 2 c] ce4x 3/3 ex 2/2. 12. y tan x 2(y 4). Separation of variables gives 2 dx. By integration, ln y 4 2 ln sin x c*. Taking exponents on both sides gives y 4 c sin2 x, y c sin2 x 4. The desired particular solution is obtained from the initial condition y(1_2) c 4 0, c 4. Answer: y 4 4 sin 2 x. 14. In (4) we have p tan x, h ln (cos x), eh 1/cos x, so that (4) gives y (cos x) [e0.01x dx c] [100 e0.01x c] cos x.cos x cos x cos x sin x dy y 4 1 x3 3 x Instructor's Manual 13 im01.qxd 9/21/05 10:17 AM Page 13 The initial condition gives y(0) 100 c 0; hence c 100. The particular solution is y 100(1 e0.01x) cos x. The factor 0.01, which we included in the exponent, has the effect that the graph of y shows a long transition period. Indeed, it takes x 460 to let the exponential function e0.01x decrease to 0.01. Choose the x-interval of the graph accordingly. 16. The standard form (1) is y y . Hence h 3 tan x, and (4) gives the general solution y e3 tan x [dx c] . To evaluate the integral, observe that the integrand is of the form 1_3(3 tan x) e3 tan x; that is, 1_3(e3 tan x). Hence the integral has the value 1_3e 3 tan x. This gives the general solution y e3 tan x [1_3e 3 tan x c] 1_3 ce 3 tan x. The initial condition gives from this y(1_4) 1_3 ce 3 4_3; hence c e 3. The answer is 1_3 e 33 tan x. 18. Bernoulli equation. First solution method: Transformation to linear form. Set y 1/u. Then y u/u2 1/u 1/u2. Multiplication by u2 gives the linear ODE in standard form u u 1. General solution u cex 1. Hence the given ODE has the general solution y 1/(cex 1). From this and the initial condition y(0) 1/(c 1) c, 2 Answer, y 1/(1 2ex). Second solution method: Separation of variables and use of partial fractions. ( ) dy dx. Integration gives ln y 1 ln y ln j j x c*. Taking exponents on both sides, we obtain 1 cex, 1 cex, y . We now continue as before. 1 1 cex 1 y 1 y y 1 y y 1 y 1 y 1 y dy (y 1) e3 tan x cos2 x 1 cos2 x 3 cos2 x 14 Instructor's Manual im01.qxd 9/21/05 10:17 AM Page 14 20. Separate variables, integrate, and take exponents: cot y dy dx/(x2 1), ln sin y arctan x c* and sin y cearctan x. Now use the initial condition y(0) 1_2: 1 ce0, c 1. Answer: y arcsin (earctan x). 22. First solution method: by setting z cos (2y) (linearization): From z we have z (2 sin 2y)y. From this, 1_2z xz 2x. This is a linear ODE. Its standard form is z 2xz 4x. In (4) this gives p 2x, h x2. Hence (4) gives the solution in terms of z in the form z ex 2 [ex2(4x) dx c] ex2[2ex2 c] 2 cex2. From this we obtain the solution y 1_2 arccos z 1_ 2 arccos (2 ce x2). Second method: Separation of variables. By algebra, y sin 2y x(cos 2y 2). Separation of variables now gives dy x dx. Integrate: 1_2 ln 2 cos 2y 1_2x 2 c*. Multiply by 2 and take exponents: ln 2 cos 2y x2 c, cos 2y 2 cex2. Solve this for y: cos 2y 2 cex 2, y 1_2 arccos (2 ce x2). 24. Bernoulli ODE. Set y u3 and note that u 3y2y. Multiply the given ODE by 3y2 to obtain 3y2y 3x2y3 ex 3 sinh x. In terms of u this gives the linear ODE u 3x2u ex 3 sinh x. In (4) we thus have h x3. The solution is u ex 3 [ex3ex3 sinh x dx c] ex3 [cosh x c] and y u1/3. sin 2y 2 cos 2y Instructor's Manual 15 im01.qxd 9/21/05 10:17 AM Page 15 For c2 this gives c2 e 3(1 1.25 0.75e2.4) 0.25e3 0.75e0.6. This gives for 3 t 6 u2 1 0.25e 3t 0.75e0.6t 1/y2. Finally, for 6 t 9 we have the ODE is u3 0.8u3 1, whose general solution is u3 1.25 c3e 0.8t. c3 is determined by the continuity condition at t 6, namely, u3(6) 1.25 c3e 4.8 u2(6) 1 0.25e 3 0.75e5.4. This gives c3 e 4.8(1.25 1 0.25e3 0.75e5.4) 0.25e4.8 0.25e1.8 0.75e0.6. Substitution gives the solution for 6 t 9: u3 1.25 (0.25e 4.8 0.25e1.8 0.75e0.6)e0.8t 1/y3. 38. Substitution gives the identity 0 0. These problems are of importance because they show why linear ODEs are preferable over nonlinear ones in the modeling process. Thus one favors a linear ODE over a nonlinear one if the model is a faithful mathematical representation of the problem. Furthermore, these problems illustrate the difference between homogeneous and nonhomogeneous ODEs. 40. We obtain (y1 y2) p(y1 y2) y1 y2 py1 py2 (y1 py1) (y2 py2) r r 0. 42. The sum satisfies the ODE with r1 r2 on the right. This is important as the key to the method of developing the right side into a series, then finding the solutions corresponding to single terms, and finally, adding these solutions to get a solution of the given ODE. For instance, this method is used in connection with Fourier series, as we shall see in Sec. 11.5. 44. (a) y Y v reduces the Riccati equation to a Bernoulli equation by removing the term h(x). The second transformation, v 1/u, is the usual one for transforming a Bernoulli equation with y2 on the right into a linear ODE. Substitute y Y 1/u into the Riccati equation to get Y u/u2 p(Y 1/u) g(Y2 2Y/u 1/u2) h. Since Y is a solution, Y pY gY2 h. There remains u/u2 p/u g(2Y/u 1/u2). Multiplication by u2 gives u pu g(2Yu 1). Reshuffle terms to get u (2Yg p)u g, the linear ODE as claimed. 18 Instructor's Manual im01.qxd 9/21/05 10:17 AM Page 18 (b) Substitute y Y x to get 1 2x4 x x4 x4 x 1, which is true. Now substitute y x 1/u. This gives 1 u/u2 (2x3 1)(x 1/u) x2(x2 2x/u 1/u2) x4 x 1. Most of the terms cancel on both sides. There remains u/u2 1/u x2/u2. Multiplication by u2 finally gives u u x2. The general solution is u cex x2 2x 2 and y x 1/u. Of course, instead performing this calculation we could have used the general formula in (a), in which 2Yg p 2x(x2) 2x3 1 1 and g x2. (c) Substitution of Y x2 shows that this is a solution. In the ODE for u you find 2Yg p 2x2 (sin x) (3 2x2 sin x) 3. Also, g sin x. Hence the ODE for u is u 3u sin x. Solution: u ce3x 0.1 cos x 0.3 sin x and y x2 1/u. (e) y y xy y/y2 by the chain rule. Hence y(x 1/y2) 0. (A) From y 0 we obtain by integrations y cx a. Substitution into the given ODE gives cx a xc 1/c; hence a 1/c. This is a family of straight lines. (B) The other factor is zero when x 1/y2. By integration, y 2x1/2 c*. Substituting y and y x1/2 into the given equation y xy 1/y, we obtain 2x1/2 c* x x1/ 2 1/x1/ 2; hence c* 0. This gives the singular solution y 2x, a curve, to which those straight lines in (A) are tangent. (f) By differentiation, 2yy y xy y 0, y(2y x) 0, (A) y 0, y cx a. By substitution, c2 xc cx a 0, a c2, y cx c2, a family of straight lines. (B) y x/2, y x2/4 c*. By substitution into the given ODE, x2/4 x2/2 x2/4 c* 0, c* 0, y x2/4, the envelope of the family; see Fig. 6 in Sec. 1.1. SECTION 1.6. Orthogonal Trajectories. Optional, page 35 Purpose. To show that families of curves F(x, y, c) 0 can be described by ODEs y ƒ(x, y) and the switch to y 1/ƒ(x, y) produces as general solution the orthogonal trajectories. This is a nice application that may also help the student to gain more self-confidence, skill, and a deeper understanding of the nature of ODEs. We leave this section optional, for reasons of time. This will cause no gap. The reason why ODEs can be applied in this fashion results from the fact that general solutions of ODEs involve an arbitrary constant that serves as the parameter of this one-parameter family of curves determined by the given ODE, and then another general solution similarly determines the one-parameter family of the orthogonal trajectories. Curves and their orthogonal trajectories play a role in several physical applications (e.g., in connection with electrostatic fields, fluid flows, and so on). Instructor's Manual 19 im01.qxd 9/21/05 10:17 AM Page 19 SOLUTIONS TO PROBLEM SET 1.6, page 36 2. xy c, and by differentiation, y xy 0; hence y y/x. The ODE of the trajectories is y x/y. By separation and integration, y2/2 x2/2 c*. Hyperbolas. (So are the given curves.) 4. By differentiation, 2yy 4x; hence y 2x/y. Thus the ODE of the trajectories is y y/2x. By separating, integrating, and taking exponents on both sides, dy/y dx/2x, ln y 1_2 ln x c**, y c*/x. 6. ye3x c. Differentiation gives (y 3y)e3x 0. Hence the ODE of the given family is y 3y. For the trajectories we obtain y 1/(3y), yy 1_3, 1_ 2y 2 1_3x c**, y 2_3x c*. 8. 2x 2yy 0, so that the ODE of the curves is y x/y. Hence the ODE of the trajectories is y y/x. Separating variables, integrating, and taking exponents gives hyperbolas as trajectories; namely, y/y 1/x, ln y ln x c**, xy c*. 10. xy1/2 ĉ, or x2y1 c. By differentiation, 2xy1 x2y2y 0, y 2y/x. This is the ODE of the given family. Hence the orthogonal trajectories have the ODE y . Thus 2yy x, y2 1_2x 2 c* (ellipses). 12. x2 y2 2cy 0. Solve algebraically for 2c: y 2c. Differentiation gives y 0. By algebra, y(1) . Solve for y: y ( ) . This is the ODE of the given family. Hence the ODE of the trajectories is y ( ) . To solve this equation, set u y/x. Then y xu u (u) .1 u 1 2 x y y x 1 2 y2 x2 2xy 2xy y2 x2 y2 x2 y2 2x y 2x y x2 y2 x2y y2 2x y x2 y x2 y2 y x 2y 20 Instructor's Manual im01.qxd 9/21/05 10:17 AM Page 20 SECTION 1.7. Existence and Uniqueness of Solutions, page 37 Purpose. To give the student at least some impression of the theory that would occupy a central position in a more theoretical course on a higher level. Short Courses. This section can be omitted. Comment on Iteration Methods Iteration methods were used rather early in history, but it was Picard who made them popular. Proofs of the theorems in this section (given in books of higher level, e.g., [A11]) are based on the Picard iteration (see CAS Project 10). Iterations are well suited for the computer because of their modest storage demand and usually short programs in which the same loop or loops are used many times, with different data. Because integration is generally not difficult for a CAS, Picard's method has gained some popularity during the past few decades. SOLUTIONS TO PROBLEM SET 1.7, page 41 2. The initial condition is given at the point x 1. The coefficient of y is 0 at that point, so from the ODE we already see that something is likely to go wrong. Separating variables, integrating, and taking exponents gives , ln y 2 ln x 1 c*, y c(x 1)2. This last expression is the general solution. It shows that y(1) 0 for any c. Hence the initial condition y(1) 1 cannot be satisfied. This does not contradict the theorems because we first have to write the ODE in standard form: y ƒ(x, y) . This shows that ƒ is not defined when x 1 (to which the initial condition refers). 4. For k 0 we still get no solution, violating the existence as in Prob. 2. For k 0 we obtain infinitely many solutions, because c remains unspecified. Thus in this case the uniqueness is violated. Neither of the two theorems is violated in either case. 6. By separation and integration, dx, ln y ln x2 4x c*. Taking exponents gives the general solution y c(x2 4x). From this we can see the answers: No solution if y(0) k 0 or y(4) k 0. A unique solution if y(x0) equals any y0 and x0 0 or x0 4. Infinitely many solutions if y(0) 0 or y(4) 0. This does not contradict the theorems because ƒ(x, y) is not defined when x 0 or 4. 2x 4 x2 4x 2x 4 x2 4x dy y 2y x 1 2 dx x 1 dy y Instructor's Manual 23 im01.qxd 9/21/05 10:17 AM Page 23 8. (A) The student should gain an understanding for the "intermediate" position of a Lipschitz condition: it is more than continuity but less than partial differentiability. (B) Here the student should realize that the linear ODE is basically simpler than a nonlinear ODE. The calculation is straightforward because ƒ(x, y) r(x) p(x)y and implies that ƒ(x, y2) ƒ(x, y1) p(x) y2 y1 My2 y1 where the boundedness p(x) M for x x0 a follows from the continuity of p in this closed interval. 10. (B) yn • • • , y e x x 1 (C) y0 1, y1 1 2x, y2 1 2x 4x 2 , • • • y(x) 1 2x 4x2 8x3 • • • (D) y (x 1)2, y 0. It approximates y 0. General solution y (x c)2. (E) y y would be a good candidate to begin with. Perhaps you write the initial choice as y0 a; then a 0 corresponds to the choice in the text, and you see how the expressions in a are involved in the approximations. The conjecture is true for any choice of a constant (or even of a continuous function of x). It was mentioned in footnote 9 that Picard used his iteration for proving his existence and uniqueness theorems. Since the integrations involved in the method can be handled on the computer quite efficiently, the method has gained in importance in numerics. SOLUTIONS TO CHAP. 1 REVIEW QUESTIONS AND PROBLEMS, page 42 12. Linear ODE. Formula (4) in Sec. 1.5 gives, since p 3, h 3x, y e3x (e3x(2x) dx c) e3x(e3x(2_3x 2_9) c) ce3x 2_3x 2_9. 14. Separate variables. y dy 16x dx, 1_2y 2 8x2 c*, y2 16x2 c. Hyperbolas. 16. Linear ODE. Standard form y xy x3 x. Use (4), Sec. 1.5, with p x, h x2/2, obtaining the general solution y ex 2/2 (ex2/2 (x3 x) dx c) ex2/2[ex2/2(x2 1) c] cex 2/2 x2 1. 18. Exact; the exactness test gives 3 sin x sinh 3y on both sides. Integrate the coefficient function of dx with respect to x, obtaining u M dx cos x cosh 3y k(y). Differentiate this with respect to y and equate the result to the coefficient function of dy: uy 3 cos x sinh 3y k(y) N. Hence k 0. 1 1 2x 8x3 3 xn1 (n 1)! x3 3! x2 2! 24 Instructor's Manual im01.qxd 9/21/05 10:17 AM Page 24 The implicit general solution is cos x cosh 3y c. 20. Solvable (A) as a Bernoulli equation or (B) by separating variables. (A) Set y2 u since a 1; hence 1 a 2. Differentiate u y2, substitute y from the given ODE, and express the resulting equation in terms of u; that is, u 2yy 2y(y 1/y) 2u 2. This is a linear ODE with unknown u. Its standard form is u 2u 2. Solve it by (4) in Sec. 1.5 or by noting that the homogeneous ODE has the general solution ce2x, and a particular solution of the nonhomogeneous ODE is 1. Hence u ce2x 1, and u y2. (B) y y 1/y (y2 1)/y, y dy/(y2 1) dx. Integrate and take exponents on both sides: 1_ 2 ln (y 2 1) x c*, y2 1 ce2x. 22. The argument of the tangent suggests to set y/x u. Then y xu, and by differentiation and use of the given ODE divided by x, y u xu tan u u; hence xu tan u. Separation of variables gives cot u du dx/x, ln sin u ln x c*, sin u cx. This yields the general solution y xu x arcsin cx. 24. We set y 2x z as indicated. Then y z 2x, y z 2 and by substitution into the given ODE, xy xz 2x z2 z 2x. Subtraction of 2x on both sides gives xz z2 z. By separation of variables and integration we obtain ( ) dz , ln j j ln x c*. We now take exponents and simplify algebraically. This yields cx, y 2x cx (y 2x 1). Solving for y, we finally have (1 cx)y 2x(1 cx) cx, y 2x 1 1/(1 cx). 26. The first term on the right suggests the substitution u y/x. Then y xu, and from the ODE, xy x(u xu) u3 xu. Subtract xu on both sides to get x2u u3. Separate variables and integrate: u3 du x2 dx, 1_2u 2 x1 c*; hence u2 . This gives the general solution y xu . x c 2/x 1 c 2/x y 2x y 2x 1 z z 1 z z 1 dx x 1 z 1 1 z dz z2 z Instructor's Manual 25 im01.qxd 9/21/05 10:17 AM Page 25 The general solution of this linear ODE is y ce0.03t 20000. The initial condition is y(0) 0 (initially no fresh air) and gives y(0) c 20000 0; hence c 20000. The particular solution of our problem is y 20000(1 e0.03t). This equals 90% if t is such that e0.03t 0.1 thus if t (ln 0.1) /(0.03) 77 [min]. 40. We use separation of variables. To evaluate the integral, we apply reduction by partial fractions. This yields [ ] dy k dx, where A and B A. By integration, A[ln y a ln y b] A ln j j kt c*. We multiply this on both sides by 1/A a b, obtaining ln j j (kt c*)(a b). We now take exponents. In doing so, we can set c ec* and have ce(ab)kt. We denote the right side by E and solve algebraically for y; then y a (y b)E, y(1 E) a bE and from the last expression we finally have y . 42. Let the tangent of such a curve y(x) at (x, y) intersect the x-axis at M and the y-axis at N, as shown in the figure. Then because of the bisection we have OM 2x, ON 2y, where O is the origin. Since the slope of the tangent is the slope y(x) of the curve, by the definition of a tangent, we obtain y ON/OM y/x. a bE 1 E y a y b y a y b y a y b 1 b a 1 a b B y b A y a dy (a y)(b y) 28 Instructor's Manual im01.qxd 9/21/05 10:17 AM Page 28 By separation of variables, integration, and taking exponents, we see that , ln y ln x c*, xy c. This is a family of hyperbolas. Section 1.7. Problem 42 x0 y N M (x, y) dx x dy y Instructor's Manual 29 im01.qxd 9/21/05 10:17 AM Page 29 CHAPTER 2 Second-Order Linear ODEs Major Changes Among linear ODEs those of second order are by far the most important ones from the viewpoint of applications, and from a theoretical standpoint they illustrate the theory of linear ODEs of any order (except for the role of the Wronskian). For these reasons we consider linear ODEs of third and higher order in a separate chapter, Chap. 3. The new Sec. 2.2 combines all three cases of the roots of the characteristic equation of a homogeneous linear ODE with constant coefficients. (In the last edition the complex case was discussed in a separate section.) Modeling applications of the method of undetermined coefficients (Sec. 2.7) follow immediately after the derivation of the method (mass–spring systems in Sec. 2.8, electric circuits in Sec. 2.9), before the discussion of variation of parameters (Sec. 2.10). The new Sec. 2.9 combines the old Sec. 1.7 on modeling electric circuits by first-order ODEs and the old Sec. 2.12 on electric circuits modeled by second-order ODEs. This avoids discussing the physical aspects and foundations twice. SECTION 2.1. Homogeneous Linear ODEs of Second-Order, page 45 Purpose. To extend the basic concepts from first-order to second-order ODEs and to present the basic properties of linear ODEs. Comment on the Standard Form (1) The form (1), with 1 as the coefficient of y, is practical, because if one starts from ƒ(x)y g(x)y h(x)y r(x), one usually considers the equation in an interval I in which ƒ(x) is nowhere zero, so that in I one can divide by ƒ(x) and obtain an equation of the form (1). Points at which ƒ(x) 0 require a special study, which we present in Chap. 5. Main Content, Important Concepts Linear and nonlinear ODEs Homogeneous linear ODEs (to be discussed in Secs. 2.12.6) Superposition principle for homogeneous ODEs General solution, basis, linear independence Initial value problem (2), (4), particular solution Reduction to first order (text and Probs. 1522) Comment on the Three ODEs after (2) These are for illustration, not for solution, but should a student ask, answers are that the first will be solved by methods in Sec. 2.7 and 2.10, the second is a Bessel equation (Sec. 5.5) and the third has the solutions c1x c2 with any c1 and c2. Comment on Footnote 1 In 1760, Lagrange gave the first methodical treatment of the calculus of variations. The book mentioned in the footnote includes all major contributions of others in the field and made him the founder of analytical mechanics. 30 im02.qxd 9/21/05 10:57 AM Page 30 Hence in (9) we have p dx dx ln 1 x2 ln j j . This gives, in terms of partial fractions, U . By integration we get the answer y2 y1u y1 U dx 1 1_2 x ln j j . The equation is Legendre's equation with parameter n 1 (as, of course, need not be mentioned to the student), and the solution is essentially a Legendre function. This problem shows the usefulness of the reduction method because it is not difficult to see that y1 x is a solution. In contrast, the power series method (the standard method) would give the second solution as an infinite series, whereas by our present method we get the solution directly, bypassing infinite series in the present special case n 1. Also note that the transition to n 2, 3, • • • is not very complicated because U depends only on the coefficient p of the ODE, which remains the same for all n, since n appears only in the last term of the ODE. Hence if we want the answer for other n, all we have to do is insert another Legendre polynomial for y1 instead of the present y1 x. 24. z (1 z2)1/2, (1 z2)1/2 dz dx, arcsinh z x c1. From this, z sinh (x c1), y cosh (x c1) c2. From the boundary conditions y(1) 0, y(1) 0 we get cosh (1 c1) c2 0 cosh (1 c1) c2. Hence c1 0 and then c2 cosh 1. The answer is (see the figure) y cosh x cosh 1. Section 2.1. Problem 24 SECTION 2.2. Homogeneous Linear ODEs with Constant Coefficients, page 53 Purpose. To show that homogeneous linear ODEs with constant coefficients can be solved by algebra, namely, by solving the quadratic characteristics equation (3). The roots may be: (Case I) Real distinct roots (Case II) A real double root ("Critical case") (Case III) Complex conjugate roots. In Case III the roots are conjugate because the coefficients of the ODE, and thus of (3), are real, a fact the student should remember. –1 –0.5 0.5 1 x –0.54 y x 1 x 1 1/2 x 1 1/2 x 1 1 x2 1 1 x2 1 x2 1 1 x2 2x 1 x2 Instructor's Manual 33 im02.qxd 9/21/05 10:57 AM Page 33 To help poorer students, we have shifted the derivation of the real form of the solutions in Case III to the end of the section, but the verification of these real solutions is done immediately when they are introduced. This will also help to a better understanding. The student should become aware of the fact that Case III includes both undamped (harmonic) oscillations (if c 0) and damped oscillations. Also it should be emphasized that in the transition from the complex to the real form of the solutions we use the superposition principle. Furthermore, one should emphasize the general importance of the Euler formula (11), which we shall use on various occasions. Comment on How to Avoid Working in Complex The average engineering student will profit from working a little with complex numbers. However, if one has reasons for avoiding complex numbers here, one may apply the method of eliminating the first derivative from the equation, that is, substituting y uv and determining v so that the equation for u does not contain u. For v this gives 2v av 0. A solution is v eax/2. With this v, the equation for u takes the form u (b 1_4a 2)u 0 and can be solved by remembering from calculus that cos x and sin x reproduce under two differentiations, multiplied by 2. This gives (9), where b 1_4a2. Of course, the present approach can be used to handle all three cases. In particular, u 0 in Case II gives u c1 c2x at once. SOLUTIONS TO PROBLEM SET 2.2, page 59 2. The standard form is y 0.7y 0.12y 0. The characteristic equation 2 0.7 0.12 (0.4)(0.3) 0 has the roots 0.4 and 0.3, so that the corresponding general solution is y1 c1e 0.4x c2e 0.3x. 4. The characteristic equation 2 4 4 2 (2)2 0 has the double root 2 , so that the corresponding general solution is y (c1 c2x)e 2 x. 6. The characteristic equation 2 2 5 (1)2 4 0 has the roots 1 2i, so that the general solution is y ex(A cos 2x B sin 2x). 8. The characteristic equation is 2 2.6 1.69 (1.3)2 0, so that we obtain the general solution y (c1 c2x)e 1.3x. 34 Instructor's Manual im02.qxd 9/21/05 10:57 AM Page 34 10. From the characteristic equation 2 2 (2)(2) 0 we see that the corresponding general solution is y c1e x2 c2ex2 12. The characteristic equation 2 2.4 4 (1.2)2 1.62 0 has the roots 1.2 1.6i. The corresponding general solution is y e1.2x(A cos 1.6x B sin 1.6x). 14. The characteristic equation 2 0.96 (0.6)(1.6) 0 has the roots 0.6 and 1.6 and thus gives the general solution y c1e 0.6x c2e 1.6x. 16. To the given basis there corresponds the characteristic equation (0.5)(3.5) 2 3 1.75 0. The corresponding ODE is y 3y 1.75y 0. 18. The characteristic equation (3) 2 3 0 gives the ODE y 3y 0. 20. We see that the characteristic equation is (1 i)(1 i) 2 2 2 0 and obtain from it the ODE y 2y 2y 0. 22. From the characteristic equation 2 2 1 (1)2 0 we obtain the general solution y (c1 c2x)e x. Its derivative is y (c2 c1 c2x)e x. Setting x 0, we obtain y(0) c1 4, y(0) c2 c1 c2 4 6, c2 2. This gives the particular solution y (4 2x)ex. 24. The characteristic equation is 10 2 50 65 10[2 5 6.5] 10[(2.5)2 0.25] 0. Hence a general solution is y e2.5x(A cos 0.5x B sin 0.5x) and y(0) A 1.5. From this we obtain the derivative y e2.5x(2.5 1.5 cos 0.5x 2.5B sin 0.5x 0.75 sin 0.5x 0.5B cos 0.5x). Instructor's Manual 35 im02.qxd 9/21/05 10:57 AM Page 35 Accordingly, differentiation of the numerator with respect to gives xe x 0, and differentiation of the denominator gives 1. The limit of this is xe x. 16. The two conditions follow trivially from the condition in the text. Conversely, by combining the two conditions we have L(cy kw) L(cy) L(kw) cLy kLw. SECTION 2.4. Modeling: Free Oscillations (Mass–Spring System), page 61 Purpose. To present a main application of second-order constant-coefficient ODEs my cy ky 0 resulting as models of motions of a mass m on an elastic spring of modulus k (0) under linear damping c (0) by applying Newton's second law and Hooke's law. These are free motions (no driving force). Forced motions follow in Sec. 2.8. This system should be regarded as a basic building block of more complicated systems, a prototype of a vibrating system that shows the essential features of more sophisticated systems as they occur in various forms and for various purposes in engineering. The quantitative agreement between experiments of the physical system and its mathematical model is surprising. Indeed, the student should not miss performing experiments if there is an opportunity, as I had as a student of Prof. Blaess, the inventor of a (now obscure) graphical method for solving ODEs. Main Content, Important Concepts Restoring force ky, damping force cy, force of inertia my No damping, harmonic oscillations (4), natural frequency 0 /(2) Overdamping, critical damping, nonoscillatory motions (7), (8) Underdamping, damped oscillations (10) SOLUTIONS TO PROBLEM SET 2.4, page 68 2. (i) k1/m/(2) 3/(2), (ii) 5/(2) (iii) Let K denote the modulus of the springs in parallel. Let F be some force that stretches the combination of springs by an amount s0. Then F Ks0. Let k1s0 F1, k2s0 F2. Then F F1 F2 (k1 k2)s0. By comparison, K k1 k2 102 [nt /m], K/m/(2) 34/(2) 5.83/(2). (iv) Let F k1s1, F k2s2. Then if we attach the springs in series, the extensions s1 and s2 under F add, so that F k(s1 s2), where k is the modulus of the combination. Substitution of s1 and s2 from the other two equations gives F k(F/k1 F/k2). Division by kF gives 1/k 1/k1 1/k2, k k1k2/(k1 k2) 19.85. Hence the frequency is f k/m/(2) 6.62/(2) 2.57/(2). 38 Instructor's Manual im02.qxd 9/21/05 10:57 AM Page 38 4. mg ks0 by Hooke's law. Hence k mg/s0, and f (1/2)k/m (1/2)mg/s0m (1/2)g/s0 (1/2)9.80/0.1. The numeric value of the last expression is 1.58 sec1, approximately; here, s0 10 cm 0.1 m is given. 6. my 0.32y, where 0.32y is the volume of water displaced when the buoy is depressed y meters from its equilibrium position, and 9800 nt is the weight of water per cubic meter. Thus y 0 2y 0, where 0 2 0.32/m and the period is 2 / 0 2; hence m 0.32 / 0 2 0.32 / 281 W mg 281 9.80 2754 [nt] (about 620 lb). 8. Team Project. (a) W ks0 25, s0 2, m W/g, and k/m (W/s0)/(W/g) 980/2 22.14. This gives the general solution y A cos 22.14t B sin 22.14t. Now y(0) A 0, y 22.14B cos 22.14t, y(0) 22.14B 15, B 0.6775. Hence the particular solution satisfying the given initial conditions is y 0.6775 sin 22.14t [cm]. (b) 0 K/I0 17.64 4.2 sec1. Hence a general solution is A cos 4.2t B sin 4.2t. The derivative is 4.2A sin 4.2t 4.2B cos 4.2t. The initial conditions give (0) A /4 0.7854 rad (45°) and (0) /12 0.2618 rad sec1 (15°sec1), hence B 0.2618/4.2 0.0623. The answer is 0.7854 cos 4.2t 0.0623 sin 4.2t. (c) The force of inertia in Newton's second law is my, where m 5 kg is the mass of the water. The dark blue portion of the water in Fig. 45, a column of height 2y, is the portion that causes the restoring force of the vibration. Its volume is 0.022 2y. Hence its weight is 0.022 2y, where 9800 nt is the weight of water per cubic meter. This gives the ODE y 0 2y 0 where 0 2 0.000 5027 4.926 and 0 2.219. Hence the corresponding general solution is y A cos 2.219t B sin 2.219t. The frequency is 0 /(2) 0.353 [sec 1], so that the water makes about 20 oscillations per minute, or one cycle in about 3 sec. 0.022 2 5 Instructor's Manual 39 im02.qxd 9/21/05 10:57 AM Page 39 10. y e2t(2 cos t sin t) 0, tan t 2, t 2.0344 n , n 0, 1, • • •. 12. If an extremum is at t0, the next one is at t1 t0 / *, by Prob. 11. Since the cosine and sine in (10) have period 2 / *, the amplitude ratio is exp(t0) /exp(t1) exp((t0 t1)) exp(/ *). The natural logarithm is / *, and maxima alternate with minima. Hence 2 / * follows. For the ODE, 2 1/(1_24 5 22) . 14. 2 / * 2 2 sec; * . The time for 15 cycles is t 30 sec. The quotient of the corresponding amplitudes at t0 and t0 30 is e(t030)/et0 e 30 0.25. Thus e30 4, (ln 4)/30 0.0462. Now c/(2m) c/4; hence c 4 0.1848. To check this, use *2 4m2 4mk c2 by (9) which gives k (4m2 *2 c2)/(4m2 2 c2) 19.74, and solve 2y 0.1848y 19.74y 0 to get y e0.0462t(A cos 3.14t B sin 3.14t) and e0.0462 30 1_4. 16. y c1e ()t c2e ()t, y(0) c1 c2 y0. By differentiating and setting t 0 it follows that y(0) ( )c1 ( )c2 v0. From the first equation, c2 y0 c1. By substitution and simplification, ( )c1 ( )(y0 c1) v0 c1( ) v0 ( )y0. This yields the answer c1 [( )y0 v0] /(2), c2 [( )y0 v0] /(2). 18. CAS Project. (a) The three cases appear, along with their typical solution curves, regardless of the numeric values of k/m, y(0), etc. (b) The first step is to see that Case II corresponds to c 2. Then we can choose other values of c by experimentation. In Fig. 46 the values of c (omitted on purpose; the student should choose!) are 0 and 0.1 for the oscillating curves, 1, 1.5, 2, 3 for the others (from below to above). (c) This addresses a general issue arising in various problems involving heating, cooling, mixing, electrical vibrations, and the like. One is generally surprised how quickly certain states are reached whereas the theoretical time is infinite. (d) General solution y(t) ect/2(A cos *t B sin *t), where * 2 1_4(4 c2). The first initial condition y(0) 1 gives A 1. For the second initial condition we need the derivative (we can set A 1) y(t) ect/2 (cos *t B sin *t * sin *t *B cos *t) .c 2 c 2 1 4m 1 4m 40 Instructor's Manual im02.qxd 9/21/05 10:57 AM Page 40 From this and the second initial condition we obtain y(1) c2 1.5c1 2.5; hence c2 2.5 1.5c1 0.8. 16. Team Project. (A) The student should realize that the present steps are the same as in the general derivation of the method in Sec. 2.1. An advantage of such specific derivations may be that the student gets a somewhat better understanding of the method and feels more comfortable with it. Of course, once a general formula is available, there is no objection to applying it to specific cases, but often a direct derivation may be simpler. In that respect the present situation resembles, for instance, that of the integral solution formula for first-order linear ODEs in Sec. 1.5. (B) The Euler–Cauchy equation to start from is x2y (1 2m s)xy m(m s)y 0 where m (1 a)/2, the exponent of the one solution we first have in the critical case. For s * 0 the ODE becomes x2y (1 2m)xy m2y 0. Here 1 2m 1 (1 a) a, and m2 (1 a)2/4, so that this is the Euler–Cauchy equation in the critical case. Now the ODE is homogeneous and linear; hence another solution is Y (xms xm) /s. L'Hôpital's rule, applied to Y as a function of s (not x, because the limit process is with respect to s, not x), gives (xm ln x) /1 * xm ln x as s * 0. This is the expected result. (C) This is less work than perhaps expected, an exercise in the technique of differentiation (also necessary in other cases). We have y xm ln x, and with (ln x) 1/x we get y mxm1 ln x xm1 y m(m 1)xm2 ln x mxm2 (m 1)xm2. Since xm x(1a) /2 is a solution, in the substitution into the ODE the ln-terms drop out. Two terms from y and one term from y remain and give x2(mxm2 (m 1)xm2) axm xm(2m 1 a) 0 because 2m 1 a. (D) t ln x, dt/dx 1/x, y y . t y . /x, where the dot denotes the derivative with respect to t. By another differentiation, y (y . /x) ÿ /x2 y . /(x2). Substitution of y and y into (1) gives the constant-coefficient ODE ÿ y . ay . by ÿ (a 1)y . by 0. Instructor's Manual 43 im02.qxd 9/21/05 10:57 AM Page 43 The corresponding characteristic equation has the roots 1_2(1 a) 1_4(1 a)2 b. With these , solutions are e t (et) (eln x) x . (E) te t (ln x)e ln x (ln x)(eln x) x ln x. SECTION 2.6. Existence and Uniqueness of Solutions. Wronskian, page 73 Purpose. To explain the theory of existence of solutions of ODEs with variable coefficients in standard form (that is, with y as the first term, not, say, ƒ(x)y) y p(x)y q(x)y 0 and of their uniqueness if initial conditions y(x0) K0, y(x0) K1 are imposed. Of course, no such theory was needed in the last section on ODEs for which we were able to write all solutions explicitly. Main Content Continuity of coefficients suffices for existence and uniqueness. Linear independence if and only if the Wronskian is not zero A general solution exists and includes all solutions. Comment on Wronskian For n 2, where linear independence and dependence can be seen immediately, the Wronskian serves primarily as a tool in our proofs; the practical value of the independence criterion will appear for higher n in Chap. 3. Comment on General Solution Theorem 4 shows that linear ODEs (actually, of any order) have no singular solutions. This also justifies the term "general solution," on which we commented earlier. We did not pay much attention to singular solutions, which sometimes occur in geometry as envelopes of one-parameter families of straight lines or curves. SOLUTIONS TO PROBLEM SET 2.6, page 77 2. y 2y 0. Wronskian W j j . 4. Auxiliary equation (m 3)(m 2) m2 m 6 m(m 1) 6 0. Hence the ODE is x2y 6y 0. The Wronskian is W j j 2 3 5. 6. Characteristic equation (3.4)(2.5) 2 0.9 8.5 0. Hence the ODE is y 0.9y 8.5y 0. Wronskian W j j 5.9e0.9x.e2.5x 2.5e2.5x e3.4x 3.4e3.4x x2 2x3 x3 3x2 sin x cos x cos x sin x 44 Instructor's Manual im02.qxd 9/21/05 10:57 AM Page 44 8. Characteristic equation (2)2 0, ODE y 4y 4y 0, Wronskian W j j e4x. 10. Auxiliary equation (m 3)2 m2 6m 9 m(m 1) 7m 9 0. Hence the ODE (an Euler–Cauchy equation) is x2y 7xy 9y 0. The Wronskian is W j j x7. 12. The characteristic equation is (2)2 2 2 4 4 2 0. Hence the ODE is y 4y (4 2)y 0. The Wronskian is W j j e4x where e1 e2x cos x, and y2 e2x sin x. 14. The auxiliary equation is (m 1)2 1 m2 2m 2 m(m 1) 3m 2 0. Hence the Euler–Cauchy equation is x2y 3xy 2y 0. The Wronskian is W j j x3 where x1 in the second row results from the chain rule and (ln x) 1/x. Here c cos (ln x), s sin (ln x). 16. The characteristic equation is (k)2 2 0. This gives the ODE y 2ky (k2 2)y 0. The Wronskian is W j j e2kx where e ekx, c cos x, s sin x. 18. Team Project. (A) c1e x c2e x c*1 cosh x c*2 sinh x. Expressing cosh and sinh in terms of exponential functions [see (17) in App. 3.1], we have 1_2(c*1 c*2)e x 1_2(c*1 c*2)e x; es e(ks c) ec e(kc s) x1 s x2s x1cx1 x1 c x2c x1sx1 es e2s c) ec e(2c s) x3 ln x 3x4 ln x x4 x3 3x4 xe2x (1 2x)e2x e2x 2e2x Instructor's Manual 45 im02.qxd 9/21/05 10:57 AM Page 45 8. 5 is a double root. 100 sinh 5x 50e5x 50e5x. Hence we may choose yp yp1 yp2 with yp1 Cxe5x yp2 Cx2e5x according to the Modification Rule. Substitution gives yp1 1_2e 5x, yp2 25x 2e5x. Answer: y (c1 c2x)e 5x 1_2e 5x 25x2e5x. 10. y e2x(A cos 1.5x B sin 1.5x) 0.5x2 0.36x 0.1096. The solution of the homogeneous ODE approaches 0, and the term in x2 becomes the dominant term. 12. Corresponding to the right side, write yp yp1 yp2. Find yp1 2x by inspection or as usual. Since sin 3x is a solution of the homogeneous ODE, write by the Modification Rule for a simple root yp x(K cos 3x M sin 3x). Answer: y A cos 3x B sin 3x 2x 6x cos 3x. 14. 2x sin x is not listed in the table because it is of minor practical importance. However, by looking at its derivatives, we see that yp Kx cos x Mx sin x N cos x P sin x should be general enough. Indeed, by substitution and collecting cosine and sine terms separately we obtain (1) (2K 2Mx 2P 2M) cos x 0 (2) (2Kx 2M 2N 2K) sin x 2x sin x. In (1) we must have 2Mx 0; hence M 0 and then P K. In (2) we must have 2Kx 2x; hence K 1, so that P 1 and from (2), finally, 2N 2K 0, hence N 1. Answer: y (c1 c2x)e x (1 x) cos x sin x. 16. y yh yp (c1 c2x)e 1.5x 12x2 20x 16. From this and the initial conditions, y 4[(1 x)e1.5x 3x2 5x 4]. 18. yh c1e 2x, c2, yp C1xe 2x C2e 2x by the Modification Rule for a simple root. Answer: y (2e2x 3 6xe2x e2x. 20. The Basic Rule and the Sum Rule are needed. We obtain y e x(A cos 3x B sin 3x) y ex cos 3x 0.4 cos x 1.8 sin x 6 cos 3x sin 3x. 22. Team Project. (b) Perhaps the simplest way is to take a specific ODE, e.g., x2y 6xy 6y r(x) and then experiment by taking various r(x) to find the form of choice functions. The simplest case is a single power of x. However, almost all the functions that work as r(x) in the case of an ODE with constant coefficients can also be used here. SECTION 2.8. Modeling: Forced Oscillations. Resonance, page 84 Purpose. To extend Sec. 2.4 from free to forced vibrations by adding an input (a driving force, here assumed to be sinusoidal). Mathematically, we go from a homogeneous to a nonhomogeneous ODE, which we solve by undetermined coefficients. 48 Instructor's Manual im02.qxd 9/21/05 10:57 AM Page 48 New Features Resonance (11) y At sin 0 t in the undamped case Beats (12) y B(cos t cos 0 t) if input frequency is close to natural Large amplitude if (15*) 2 0 2 c2/(2m2) (Fig. 56) Phase lag between input and output SOLUTIONS TO PROBLEM SET 2.8, page 90 2. yp 0.6 cos 1.5t 0.2 sin 1.5t. Note that a general solution of the homogeneous ODE is yh e t(A cos 1.5t B sin 1.5t), and the student should perhaps be reminded that this is not resonance, of course. 4. yp 0.25 cos t. Note that yh e 2t(A cos t B sin t); of course, this is not resonance. Furthermore, it is interesting that whereas a single term on the right will generate two terms in the solution, here we have—by chance—the converse. 6. yp 1_10 cos t 1_5 sin t 1_90 cos 3t 1_45 sin 3t 8. yp 2 cos 4t 1.5 sin 4t 10. y (c1 c2t)e 2t 0.03 cos 4t 0.04 sin 4t 12. y c1e t c2e 4t 0.1 cos 2t. Note that, ordinarily, yp will consist of two terms if r(x) consists of a single trigonometric term. 14. y et(A cos 2t B sin 2t) 0.2 cos t 0.1 cos t 0.2 sin t 16. y 1_163 cos 8t 1_8 sin 8t 1_63 cos t. From the graph one can see the effect of (cos t) /63. There is no corresponding sine term because there is no damping and hence no phase shift. 18. y (33 31t)et 37.5 cos t 6 cos 2t 4.5 sin 2t 1.5 cos 3t 2 sin 3t 20. y 100 cos 4.9t 98 cos 5t 22. The form of solution curves varies continuously with c. Hence if you start from c 0 and let c increase, you will at first obtain curves similar to those in the case of c 0. For instance, consider y 0.01y 25y 100 cos 4.9t 98 cos 5t. 24. CAS Experiment. The choice of needs experimentation, inspection of the curves obtained, and then changes on a trial-and-error basis. It is interesting to see how in the case of beats the period gets increasingly longer and the maximum amplitude gets increasingly larger as /(2) approaches the resonance frequency. 26. If 0 t , then a particular solution yp K0 K1t K2t 2 gives yp 2K2 and yp yp K0 2K2 K1t K2t 2 1 t2; thus, K2 , K1 0, K0 1 2K2 1 . Hence a general solution is y A cos t B sin t 1 t2. 1 2 2 2 2 1 2 1 2 Instructor's Manual 49 im02.qxd 9/21/05 10:57 AM Page 49 From this and the first initial condition, y(0) A 1 0, A (1) . The derivative is y A sin t B cos t and gives y(0) B 0. Hence the solution is (I) y(t) (1 2/ 2)(1 cos t) t2/ 2 if 0 t , and if t , then (II) y A2 cos t B2 sin t with A2 and B2 to be determined from the continuity conditions y( ) y2( ), y( ) y2( ). So we need from (I) and (II) y( ) 2(1 2/ 2) 1 1 4/ 2 A2 and y(t) (1 2/ 2) sin t 2t / 2 and from this and (II), y( ) 2/ B cos B2. This gives the solution y (1 4/ 2) cos t (2/) sin t if t . Answer: y { . The function in the second line gives a harmonic oscillation because we disregarded damping. SECTION 2.9. Modeling: Electric Circuits, page 91 Purpose. To discuss the current in the RLC-circuit with sinusoidal input E0 sin t. ATTENTION! The right side in (1) is E0 cos t, because of differentiation. Main Content Modeling by Kirchhoff's law KVL Electrical–mechanical strictly quantitative analogy (Table 2.2) Transient tending to harmonic steady-state current A popular complex method is discussed in Team Project 20. if 0 t if t (1 2/ 2)(1 cos t) t2/ 2 (1 4/ 2) cos t (2/) sin t 2 2 2 2 50 Instructor's Manual im02.qxd 9/21/05 10:57 AM Page 50 The real part of Kei t is (Re K)(Re ei t) cos t sin t (S cos t R sin t), in agreement with (2) and (4). (b) See (A). (c) R 2 , L 1 H, C 1_3 F, 1, S 1 3 2, E0 20. From this and (4) it follows that a 5, b 5; hence Ip 5 cos t 5 sin t. For the complex method we obtain from (A) K 5 5i. Hence Ip Re(Ke it) Re[(5 5i)(cos t i sin t)] 5 cos t 5 sin t. SECTION 2.10. Solution by Variation of Parameters, page 98 Purpose. To discuss the general method for particular solutions, which applies in any case but may often lead to difficulties in integration (which we by and large have avoided in our problems, as the subsequent solutions show). Comments The ODE must be in standard form, with 1 as the coefficient of y—students tend to forget that. Here we do need the Wronskian, in contrast with Sec. 2.6 where we could get away without it. SOLUTIONS TO PROBLEM SET 2.10, page 101 2. y1 e 2x, y2 xe 2x, W e4x, yh (c1 c2x)e 2x, yp e 2xx3ex dx xe2xx2ex dx (x2 4x 6)ex. 4. yh (c1 c2x)e x, yp e x(A cos x B sin x), A 0 from the cosine terms, B 1 from the sine terms, so that yp e x sin x. 6. Division by x2 gives the standard form, and r x1 ln x. A basis of solutions is y1 x, y2 x ln x; W x. The corresponding particular solution obtained from (2) is yp x(ln x)x(1 ln x)x1 dx (x ln x)x(x1 ln x)x1 dx x(ln x)2 x1dx x ln x(ln x)/x dx x(ln x)3/6. 20(2 2i) 8 20 2 2i E0 S2 R2 E0R S2 R2 Instructor's Manual 53 im02.qxd 9/21/05 10:57 AM Page 53 8. yh (c1 c2x)e 2x, y1 e 2x, y2 xe 2x, W e4x. From (2) we thus obtain yp 12e 2xx3 dx 12xe2xx4 dx 2x2e2x. 10. The right side suggests the following choice of a basis of solutions: y1 cosh x, y2 sinh x. Then W 1, and yp cosh x(sinh x) /cosh x dx sinh x(cosh x) /cosh x dx x sinh x. 12. Divide by x to get the standard form with r 3x3 3x1. A basis of homogeneous solutions is y1 x 1/2, y2 x 1/2. The Wronskian is W x1. From this and (2) we obtain yp x 1/2x1/2(3x3 3x1)(x) dx x1/2x1/2(3x3 3x1)(x) dx 4x1 4x. 14. y1 x 2, y2 x 2, W 4x1. Hence (2) gives yp x 2x2x4(x/4) dx x2x2x4(x/4) dx 1_4x 2 ln x 1_116x 2. 16. y ux1/2 leads to u 0 by substitution. (This is a special case of the method of elimination of the first derivative, to be discussed in general in Prob. 29 of Problem Set 5.5 on the Bessel equation.) Hence a basis of solutions for which the Bessel functions reduce to elementary functions, namely, to cosines and sines times powers of x.) Hence a basis of solutions of the homogeneous ODE corresponding to the given ODE is y1 x 1/2 cos x, y2 x 1/2 sin x. The Wronskian is W x1 and r x1/2 cos x. Hence (2) gives yp x 1/2 cos xx1/2 x1/2 (cos x) x dx x1/2 sin xx1/2 (cos x) xx1/2 x dx x1/2 sin xcos x 4x sin x 1_2x 1/2 cos x. Note here that we have used that the ODE must be in standard form before we can apply (2). This is similar to the case of the Euler–Cauchy equations in this problem set. 18. Team Project. (a) Undetermined coefficients: Substitute yp A cos 5x B sin 5x yp 5A sin 5x 5B cos 5x yp 25A cos 5x 25B sin 5x. 54 Instructor's Manual im02.qxd 9/21/05 10:57 AM Page 54 The cosine terms give 25A 10B 15A 0; hence B 4A. The sine terms give 25B 10A 15B 170A 17; hence A 0.1, B 0.4. For the method of variation of parameters we need y1 e 3x, y2 e 5x; hence W 8e2x. From this and formula (2) we obtain integrals that are not too pleasant to evaluate, namely, yp e 3xe5x 17 sin 5x (e2x/8) dx e5xe3x 17 sin 5x (e2x/8) dx 0.1 cos 5x 0.4 sin 5x. (b) Apply variation of parameters to the first term, yp1 sec 3x, using y1 cos 3x, y2 sin 3x, and W 3. Formula (2) gives yp1 cos 3x(sin 3x sec 3x) /3 dx sin 3x(cos 3x sec 3x) /3 dx 1_9 cos 3x ln cos 3x 1_ 3x sin 3x. For yp2 the method of undetermined coefficients gives yp2 1_ 6x cos 3x. SOLUTIONS TO CHAP. 2 REVIEW QUESTIONS AND PROBLEMS, page 102 10. The characteristic equation is 3 is a double root of the characteristic equation of the homogeneous ODE, so that the Modification Rule applies. The second term on the right, 27x2, requires a quadratic polynomial. A general solution is y (c1 c2x)e 3x 1_2e 3x 3x2 4x 2. 12. y z, y (dz /dy)z by the chain rule, yz dz/dy 2z2, dz /z 2 dy/y, ln z 2 ln y c*, z c1y2 y, dy/y2 c1 dx, 1/y c1x c2; hence y 1/(c1x c2). Also, y 0 is a solution. 14. y1 x 2, y2 x 3, W x6, r 1 because to apply (2), one must first cast the given ODE into standard form. Then (2) gives yp x 2x3 1(x6) dx x3x2 1(x6) dx x2(1_4 1_5) 1_20x2. 16. y1 e 2x cos x, y2 e 2x sin x, W e4x, so that (2) gives yp e 2x cos xe2x sin x e2x csc x e4x dx e2x sin xe2x cos x e2x csc x e4x dx e2x (cos x) x e2x (sin x) ln sin x. Instructor's Manual 55 im02.qxd 9/21/05 10:57 AM Page 55 34. The real ODE is 0.4I 40I 10000I 220 314 cos 314t. The complex ODE is 0.4I 40I 10000I 220 314e314it. Substitution of I Ke314it, I 314iKe314it, I 3142Ke314it into the complex ODE gives (0.4(3142) 40 314i 10000)Ke314it 220 314e314it. Solve this for K. Denote the expression (• • •) on the left by M. Then K 1.985219 0.847001i. Furthermore, the desired particular solution Ip of the real ODE is the real part of Ke314it; that is, Re(Ke314it) Re K Re(e314it) Im K Im(e314it) 1.985219 cos 314t (0.847001) sin 314t. 220 314 M 58 Instructor's Manual im02.qxd 9/21/05 10:57 AM Page 58 CHAPTER 3 Higher Order Linear ODEs This chapter is new. Its material is a rearranged and somewhat extended version of material previously contained in some of the sections of Chap 2. The rearrangement is such that the presentation parallels that in Chap. 2 for second-order ODEs, to facilitate comparisons. Root Finding For higher order ODEs you may need Newton's method or some other method from Sec. 19.2 (which is independent of other sections in numerics) in work on a calculator or with your CAS (which may give you a root-finding method directly). Linear Algebra The typical student may have taken an elementary linear algebra course simultaneously with a course on calculus and will know much more than is needed in Chaps. 2 and 3. Thus Chaps. 7 and 8 need not be taken before Chap. 3. In particular, although the Wronskian becomes useful in Chap. 3 (whereas for n 2 one hardly needs it), a very modest knowledge of determinants will suffice. (For n 2 and 3, determinants are treated in a reference section, Sec. 7.6.) SECTION 3.1. Homogeneous Linear ODEs, page 105 Purpose. Extension of the basic concepts and theory in Secs. 2.1 and 2.6 to homogeneous linear ODEs of any order n. This shows that practically all the essential facts carry over without change. Linear independence, now more involved as for n 2, causes the Wronskian to become indispensable (whereas for n 2 it played a marginal role). Main Content, Important Concepts Superposition principle for the homogeneous ODE (2) General solution, basis, particular solution General solution of (2) with continuous coefficients exists. Existence and uniqueness of solution of initial value problem (2), (5) Linear independence of solutions, Wronskian General solution includes all solutions of (2). Comment on Order of Material In Chap. 2 we first gained practical experience and skill and presented the theory of the homogeneous linear ODE at the end of the discussion, in Sec. 2.6. In this chapter, with all the experience gained on second-order ODEs, it is more logical to present the whole theory at the beginning and the solution methods (for linear ODEs with constant coefficients) afterward. Similarly, the same logic applies to the nonhomogeneous linear ODE, for which Sec. 3.3 contains the theory as well as the solution methods. SOLUTIONS TO PROBLEM SET 3.1, page 111 2. Problems 1–5 should give the student a first impression of the changes occurring in the transition from n 2 to general n. 59 im03.qxd 9/21/05 11:04 AM Page 59 8. Let y1 x 1, y2 x 2, y3 x. Then y2 2y1 y3 0 shows linear dependence. 10. Linearly independent 12. Linear dependence, since one of the functions is the zero function 14. cos 2x cos2 x sin2 x; linearly dependent 16. (x 1)2 (x 1)2 4x 0; linearly dependent 18. Linearly independent. 20. Team Project. (a) (1) No. If y1 0, then (4) holds with any k1 0 and the other kj all zero. (2) Yes. If S were linearly dependent on I, then (4) would hold with a kj 0 on I, hence also on J, contradicting the assumption. (3) Not necessarily. For instance, x2 and xx are linearly dependent on the interval 0 x 1, but linearly independent on 1 x 1. (4) Not necessarily. See the answer to (3). (5) Yes. See the answer to (2). (6) Yes. By assumption, k1y1 • • • kpyp 0 with k1, • • • , kp not all zero (this refers to the functions in S), and for T we can add the further functions with coefficients all zero; then the condition for linear dependence of T is satisfied. (b) We can use the Wronskian for testing linear independence only if we know that the given functions are solutions of a homogeneous linear ODE with continuous coefficients. Other means of testing are the use of functional relations, e.g., ln x2 2 ln x or trigonometric identities, or the evaluation of the given functions at several values of x, to see whether we can discover proportionality. SECTION 3.2. Homogeneous Linear ODEs with Constant Coefficients, page 111 Purpose. Extension of the algebraic solution method for constant-coefficient ODEs from n 2 (Sec. 2.2) to any n, and discussion of the increased number of possible cases: Real different roots Complex simple roots Real multiple roots Complex multiple roots Combinations of the preceding four basic cases Explanation of these cases in terms of typical examples Comment on Numerics In practical cases, one may have to use Newton's method or another method for computing (approximate values of) roots in Sec. 19.2. SOLUTIONS TO PROBLEM SET 3.2, page 115 2. The form of the given functions shows that the characteristic equation has a triple root 2; hence it is (2)3 3 62 12 8 0. 60 Instructor's Manual im03.qxd 9/21/05 11:04 AM Page 60 Hence a general solution of the homogeneous ODE is yh c1e x c2e 2x c3e 3x. Since both terms on the right are solutions of the homogeneous ODE, the Modification Rule for undetermined coefficients applies. Substitution into the nonhomogeneous ODE gives the particular solution yp (7 10x)e 3x (1_2 3x)e x. 6. The homogeneous Euler–Cauchy equation can be solved as usual by substituting xm. The auxiliary equation has the roots 2, 0, 1; hence a general solution is yh c1x 2 c2 c3x. This result can also be obtained by separating variables and integrating twice; that is, y/y 4/x, ln y 4 ln x c, y c1x 4, and so on. Variation of parameters gives yp 8e x(x1 x2). 8. The characteristic equation of the homogeneous ODE 3 22 9 18 0 has the roots 3, 2, and 3. Hence a general solution of the homogeneous ODE is yh c1e 3x c2e 2x c3e 3x. This also shows that the function on the right is a solution of the homogeneous ODE. Hence the Modification Rule applies, and the particular solution obtained is yp 0.2xe 2x. 10. The characteristic equation of the homogeneous ODE is 4 16 (2 4)(2 4) 0. Hence a general solution of the homogeneous ODE is yh c1e 2x c2e 2x c3 cos 2x c4 sin 2x. A particular solution yp of the nonhomogeneous ODE can be obtained by the method of undetermined coefficients; yp can be written in terms of exponential or hyperbolic functions: yp 1.5(e 2x e2x) 2x(e2x e2x) 3 cosh 2x 4x sinh 2x. Applying the initial conditions, we obtain the answer y 4e2x 16 sin 2x yp, from which we cannot immediately recognize a basis of solutions. Of course, when sin 2x occurs, cos 2x must be in the basis. But e2x remains hidden and cannot be seen from the answer. 12. The characteristic equation is quadratic in 2, the roots being 5, 1, 1, 5. The right side requires a quadratic polynomial whose coefficients can be determined by Instructor's Manual 63 im03.qxd 9/21/05 11:04 AM Page 63 substitution. Finally, the initial conditions are used to determine the four arbitrary constants in the general solution of the nonhomogeneous ODE thus obtained. The answer is y 5ex e5x 6.16 4x 2x2. Again, two of four possible terms resulting from the homogeneous ODE are not visible in the answer. The student should recognize that all or some or none of the solutions of a basis of the homogeneous ODE may be present in the final answer; this will depend on the initial conditions, so the student should experiment a little with this problem to see what is going on. 14. The method of undetermined coefficients gives yp 0.08 cos x 0.04 sin x. A basis of solutions of the homogeneous ODE is e2x, e2x, ex/2. From this and the initial conditions we obtain the answer y 0.11e2x 0.15e2x 0.96ex/2 yp in which all three basis functions occur. 16. The first equation has as a general solution y (c1 c2x c3x 2)e4x _8105x 7/2e4x. Hence in cases such as this one, one can try yp x 1/2(a0 a1x a2x 2 a3x 3)e4x. One can now modify the right side systematically and see how the solution changes. The second ODE has as a general solution y c1x 2 c2x c3x 3 _1216x(18(ln x) 2 6 ln x 7). This shows that undetermined coefficients would not be suitable—the function on the right gives way to what yp may look like. Of course, the dependence on the left side also remains to be explored. SOLUTIONS TO CHAP. 3 REVIEW QUESTIONS AND PROBLEMS, page 122 6. The characteristic equation is 3 62 18 40 [(1)2 9](4) 0. Hence a general solution is y c1e 4x ex(c2 cos 3x c3 sin 3x). 8. The characteristic equation is quadratic in 2; namely, (2 1)(2 9) 0. Hence a general solution is y c1 cos x c2 sin x c3 cos 3x c4 sin 3x. 10. The characteristic equation has the triple root 1 because it is (1)3 0. 64 Instructor's Manual im03.qxd 9/21/05 11:04 AM Page 64 Hence a general solution of the homogeneous ODE is yh (c1 c2x c3x 2)e x. The method of undetermined coefficients gives the particular solution yp x 2 6x 12. 12. The characteristic equation is 2(2)(4) 0. Hence a general solution of the homogeneous ODE is yh c1 c2e 2x c3e 4x c4e 2x. The method of undetermined coefficients gives the particular solution of the nonhomogeneous ODE in the form yp 0.3 cos 2x 0.1 sin 2x. 14. The auxiliary equation of the homogeneous ODE x3y ax2y bxy cy 0 is m(m 1)(m 2) am(m 1) bm c m3 (a 3)m2 (b a 2)m c 0. In our equation, a 3, b 6, and c 6. Accordingly, the auxiliary equation becomes m3 6m2 11m 6 (m 1)(m 2)(m 3) 0. Hence a general solution of the homogeneous ODE is yh c1x c2x 2 c3x 3. Variation of parameters gives the particular solution yp 0.5x 2. 16. The characteristic equation of the ODE is 3 22 4 8 (2)(2 4) 0. Hence a general solution is yh c1e 2x c2 cos 2x c3 sin 2x. Using the initial conditions, we obtain the particular solution y 3e2x 4 cos 2x 12 sin 2x. 18. The characteristic equation of the homogeneous ODE is (2 25) 0. Hence a general solution of the homogeneous ODE is yh c1 c2 cos 5x c3 sin 5x. Instructor's Manual 65 im03.qxd 9/21/05 11:04 AM Page 65 SECTION 4.0. Basics of Matrices and Vectors, page 124 Purpose. This section is for reference and review only, the material being restricted to what is actually needed in this chapter, to make it self-contained. Main Content Matrices, vectors Algebraic matrix operations Differentiation of vectors Eigenvalue problems for 2 2 matrices Important Concepts and Facts Matrix, column vector and row vector, multiplication Linear independence Eigenvalue, eigenvector, characteristic equation Some Details on Content Most of the material is explained in terms of 2 2 matrices, which play the major role in Chap. 4; indeed, n n matrices for general n occur only briefly in Sec. 4.2 and at the beginning in Sec. 4.3. Hence the demand of linear algebra on the student in Chap. 4 will be very modest, and Sec. 4.0 is written accordingly. In particular, eigenvalue problems lead to quadratic equations only, so that nothing needs to be said about difficulties encountered with 3 3 or larger matrices. Example 1. Although the later sections include many eigenvalue problems, the complete solution of such a problem (the determination of the eigenvalues and corresponding eigenvectors) is given in Sec. 4.0. SECTION 4.1. Systems of ODEs as Models, page 130 Purpose. In this section the student will gain a first impression of the importance of systems of ODEs in physics and engineering and will learn why they occur and why they lead to eigenvalue problems. Main Content Mixing problem Electrical network Conversion of single equations to systems (Theorem 1) The possibility of switching back and forth between systems and single ODEs is practically quite important because, depending on the situation, the system or the single ODE will be the better source for obtaining the information sought in a specific case. Background Material. Secs. 2.4, 2.8. Short Courses. Take a quick look at Sec. 4.1, skip Sec. 4.2 and the beginning of Sec. 4.3, and proceed directly to solution methods in terms of the examples in Sec. 4.3. Some Details on Content Example 1 extends the physical system in Sec. 1.3, consisting of a single tank, to a system of two tanks. The principle of modeling remains the same. The problem leads to a typical eigenvalue problem, and the solutions show typical exponential increases and decreases. 68 Instructor's Manual im04.qxd 9/21/05 11:08 AM Page 68 Example 2 leads to a nonhomogeneous first-order system (a kind of system to be considered in Sec. 4.6). The vector g on the right in (5) causes the term 3 in I1 but has no effect on I2, which is interesting to observe. If time permits, one could add a little discussion of particular solutions corresponding to different initial conditions. Reduction of single equations to systems (Theorem 1) should be emphasized. Example 3 illustrates it, and further applications follow in Sec. 4.5. It helps to create a "uniform" theory centered around first-order systems, along with the possibility of reducing higher order systems to first order. SOLUTIONS TO PROBLEM SET 4.1, page 135 2. The two balance equations (Inflow minus Outflow) change to y1 0.004y2 0.02y1 y2 0.02y1 0.004y2 where 0.004 appears because we divide through the content of the new tank, which is five times that of the old T2. Ordering the system by interchanging the two terms on the right in the first equation and writing the system as a vector equation, we have y Ay, where A [ ]. The characteristic polynomial is 2 0.024 (0.024). Hence the eigenvalues are 0 (as before) and 0.024. Eigenvectors can be obtained from the first component of the vector equation Ax x; that is, 0.02x1 0.004x2 x1. For 1 0 this is 0.02x1 0.004x2, say, x1 1, x2 5. For 2 0.024 this is 0.02x1 0.004x2 0.024x1. This simplifies to 0.004x1 0.004x2 0. A solution is x1 1, x2 1. Hence a general solution of the system of ODEs is y c1 [ ] c2 [ ]e0.024t. For t 0 this becomes, using the initial conditions y1(0) 0, y2(0) 150, y(0) [ ] [ ]. Solution: c1 25, c2 25. This gives the particular solution y 25 [ ] 25 [ ]e0.024t. The situation described in the answer to Example 1 can no longer be achieved with the new tank, because the limits are 25 lb and 125 lb, as the particular solution shows. 1 1 1 5 0 150 c1 c2 5c1 c2 1 1 1 5 0.004 0.004 0.02 0.02 Instructor's Manual 69 im04.qxd 9/21/05 11:08 AM Page 69 4. With a we can write the system that models the process in the following form: y1 ay2 ay1 y2 ay1 ay2, ordered as needed for the proper vector form y1 ay1 ay2 y2 ay1 ay2. In vector form, y Ay, where A [ ]. The characteristic equation is (a)2 a2 2 2a 0. Hence the eigenvalues are 0 and 2a. Corresponding eigenvectors are [ ] and [ ] respectively. The corresponding "general solution" is y c1 [ ] c2 [ ]e2at. This result is interesting. It shows that the solution depends only on the ratio a, not on the tank size or the flow rate alone. Furthermore, the larger a is, the more rapidly y1 and y2 approach their limit. The term "general solution" is in quotation marks because this term has not yet been defined formally, although it is clear what is meant. 6. The matrix of the system is A Y Z . The characteristic polynomial is 3 0.082 0.0012 (0.02)(0.06). This gives the eigenvalues and corresponding eigenvectors 1 0, x (1) Y Z , 2 0.02, x(2) Y Z , 3 0.06, x(3) Y Z . 1 2 1 1 0 1 1 1 1 0 0.02 0.02 0.02 0.04 0.02 0.02 0.02 0 1 1 1 1 1 1 1 1 a a a a Flow rate Tank size 70 Instructor's Manual im04.qxd 9/21/05 11:08 AM Page 70 The characteristic equation is 2 15 50 (5)(10) 0. The roots are 5 and 10. Eigenvectors are x(1) [ ] and x(2) [ ]. The corresponding general solution is y c1 [ ] e5t c2 [ ] e10t. 16. Team Project. (a) From Sec. 2.4 we know that the undamped motions of a mass on an elastic spring are governed by my ky 0 or my ky where y y(t) is the displacement of the mass. By the same arguments, for the two masses on the two springs in Fig. 80 we obtain the linear homogeneous system (11) for the unknown displacements y1 y1(t) of the first mass m1 and y2 y2(t) of the second mass m2. The forces acting on the first mass give the first ODE, and the forces acting on the second mass give the second ODE. Now m1 m2 1, k1 12, and k2 8 in Fig. 80 so that by ordering (11) we obtain or, written as a single vector equation, y [ ] [ ][ ] . (b) As for a single equation, we try an exponential function of t, y xe t. Then y 2xe t Axe t. Writing 2 and dividing by e t, we get Ax x. Eigenvalues and eigenvectors are 1 4, x (1) [ ] and 2 24, x(2) [ ] . Since and 4 2i and 24 i24, we get y x(1)(c1e 2it c2e 2it) x(2)(c3e i24 t c4ei24 t) or, by (10) in Sec. 2.2, y a1x (1) cos 2t b1x (1) sin 2t a2x (2) cos 24 t b2x(2) sin 24 t 2 1 1 2 y1 y2 8 8 20 8 y1 y2 20y1 8y2 8y1 8y2 y1 y2 m1y1 k1y1 k2(y2 y1) m2y2 k2(y2 y1) 1 10 1 5 1 10 1 5 Instructor's Manual 73 im04.qxd 9/21/05 11:08 AM Page 73 where a1 c1 c2, b1 i(c1 c2), a2 c3 c4, b2 i(c3 c4). These four arbitrary constants can be specified by four initial conditions. In components, this solution is y1 a1 cos 2t b1 sin 2t 2a2 cos 24 t 2b2 sin 24 t y2 2a1 cos 2t 2b1 sin 2t a2 cos 24 t b2 sin 24 t. (c) The first two terms in y1 and y2 give a slow harmonic motion, and the last two a fast harmonic motion. The slow motion occurs if at some instant both masses are moving downward or both upward. For instance, if a1 1 and all other arbitrary constants are zero, we get y1 cos 2t, y2 2 cos 2t; this is an example of such a motion. The fast motion occurs if at each instant the two masses are moving in opposite directions, so that one of the two springs is extended, whereas the other is simultaneously compressed. For instance, if a2 1 and all other constants are zero, we have y1 2 cos 24 t, y2 cos 24 t; this is a fast motion of the indicated type. Depending on the initial conditions, one or the other motion will occur, or a superposition of both. SECTION 4.2. Basic Theory of Systems of ODEs, page 136 Purpose. This survey of some basic concepts and facts on nonlinear and linear systems is intended to give the student an impression of the conceptual and structural similarity of the theory of systems to that of single ODEs. Content, Important Concepts Standard form of first-order systems Form of corresponding initial value problems Existence of solutions Basis, general solution, Wronskian Background Material. Sec. 2.6 contains the analogous theory for single equations. See also Sec. 1.7. Short Courses. This section may be skipped, as mentioned before. SECTION 4.3. Constant-Coefficient Systems. Phase Plane Method, page 139 Purpose. Typical examples show the student the rich variety of pattern of solution curves (trajectories) near critical points in the phase plane, along with the process of actually solving homogeneous linear systems. This will also prepare the student for a good understanding of the systematic discussion of critical points in the phase plane in Sec. 4.4. Main Content Solution method for homogeneous linear systems Examples illustrating types of critical points Solution when no basis of eigenvectors is available (Example 6) Important Concepts and Facts Trajectories as solution curves in the phase plane Phase plane as a means for the simultaneous (qualitative) discussion of a large number of solutions Basis of solutions obtained from basis of eigenvectors 74 Instructor's Manual im04.qxd 9/21/05 11:08 AM Page 74 Background Material. Short review of eigenvalue problems from Sec. 4.0, if needed. Short Courses. Omit Example 6. Some Details on Content In addition to developing skill in solving homogeneous linear systems, the student is supposed to become aware that it is the kind of eigenvalues that determine the type of critical point. The examples show important cases. (A systematic discussion of all cases follows in the next section.) Example 1. Two negative eigenvalues give a node. Example 2. A real double eigenvalue gives a node. Example 3. Real eigenvalues of opposite sign give a saddle point. Example 4. Pure imaginary eigenvalues give a center, and working in complex is avoided by a standard trick, which can also be useful in other contexts. Example 5. Genuinely complex eigenvalues give a spiral point. Some work in complex can be avoided, if desired, by differentiation and elimination. The first ODE is (a) y2 y1 y1. By differentiation and from the second ODE as well as from (a), y1 y1 y2 y1 y1 (y1 y1) 2y1 2y1. Complex solutions e(1 i)t give the real solution y1 e t(A cos t B sin t). From this and (a) there follows the expression for y2 given in the text. Example 6 shows that the present method can be extended to include cases when A does not provide a basis of eigenvectors, but then becomes substantially more involved. In this way the student will recognize the importance of bases of eigenvectors, which also play a role in many other contexts. SOLUTIONS TO PROBLEM SET 4.3, page 146 2. The eigenvalues are 5 and 5. Eigenvectors are [1 1]T and [1 1]T, respectively. Hence a general solution is y1 c1e 5t c2e 5t y2 c1e 5t c2e 5t. 4. The eigenvalues are 13.5 and 4.5. Eigenvectors are [3 1]T and [3 1]T, respectively. Hence a general solution is y1 3c1e 13.5t 3c2e 4.5t y2 c1e 13.5t c2e 4.5t. 6. The eigenvalues are complex, 2 2i and 2 2i. Corresponding complex eigenvectors are [1 i]T and [1 i]T, respectively. Hence a complex general solution is y1 c1e (22i)t c2e (22i)t y2 ic1e (22i)t ic2e (22i)t. Instructor's Manual 75 im04.qxd 9/21/05 11:08 AM Page 75 For Tank T2 it is y2 y1 y2. Performing the divisions and ordering terms, we have The eigenvalues of the matrix of this system are 0.24 and 0.08. Eigenvectors are [1 2]T and [1 2]T, respectively. The corresponding general solution is y c1[ ]e0.24t c2[ ]e0.08t. The initial conditions are y1(0) 100, y2(0) 40. This gives c1 40, c2 60. In components the answer is y1 40e 0.24t 60e0.08t y2 80e 0.24t 120e0.08t. Both functions approach zero as t * , a reasonable result, because pure water flows in and mixture flows out. SECTION 4.4. Criteria for Critical Points. Stability, page 147 Purpose. Systematic discussion of critical points in the phase plane from the standpoints of both the geometrical shapes of trajectories and stability. Main Content Table 4.1 for the types of critical points Table 4.2 for the stability behavior Stability chart (Fig. 91), giving Tables 4.1 and 4.2 graphically Important Concepts Node, saddle point, center, spiral point Stable and attractive, stable, unstable Background Material. Sec. 2.4 (needed in Example 2). Short Courses. Since all these types of critical points already occurred in the previous section, one may perhaps present just a short discussion of stability. Some Details on Content The types of critical points in Sec. 4.3 now recur, and the discussion shows that they exhaust all possibilities. With the examples of Sec. 4.3 fresh in mind, the student will acquire a deeper understanding by discussing the stability chart and by reconsidering those examples from the viewpoint of stability. This gives the instructor an opportunity to emphasize that the general importance of stability in engineering can hardly be overestimated. Example 2, relating to the familiar free vibrations in Sec. 2.4, gives a good illustration of stability behavior, namely, depending on c, attractive stability, stability (and instability if one includes "negative damping," with c 0, as it will recur in the next section in connection with the famous van der Pol equation). 1 2 1 2 0.04y2 0.16y2. 0.16y1 0.16y1 y1 y2 16 48 400 64 400 78 Instructor's Manual im04.qxd 9/21/05 11:08 AM Page 78 SOLUTIONS TO PROBLEM SET 4.4, page 150 2. p 7, q 12 0, 49 48 0, unstable node, y1 c1e 4t, y2 c2e 3t 4. p 2, q 5, 4 20 0, stable and attractive spiral. The components of a general solution are y1 c1e (12i)t c2e (12i)t et((c1 c2) cos 2t i(c1 c2) sin 2t) et(A cos 2t B sin 2t), y2 (1 2i)c1e (12i)t (1 2i)c2e (12i)t et((c1 c2 2ic1 2ic2) cos 2t i(2ic1 2ic2 c1 c2) sin 2t) et((A 2B) cos 2t (2A B) sin 2t) where A c1 c2 and B i(c1 c2). 6. p 7, q 78, saddle point, unstable, y1 c1e 13t c2e 6t y2 1.4c1e 13t 0.5c2e 6t 8. p 0, q 16, center, stable, eigenvalues 4i, 4i, eigenvectors [1 0.6 0.8i]T and [1 0.6 0.8i]T, respectively, solution: y1 c1e 4it c2e 4it A cos 4t B sin 4t y2 (0.6 0.8i)c1e 4it (0.6 0.8i)c2e 4it (0.6 0.8i)c1(cos 4t i sin 4t) (0.6 0.8i)c2(cos 4t i sin 4t) (0.6(c1 c2) 0.8i(c1 c2)) cos 4t (0.6i(c1 c2) 0.8(c1 c2)) sin 4t (0.6A 0.8B) cos 4t (0.6B 0.8A) sin 4t where A c1 c2 and B i(c1 c2). 10. y1 y c1e 5t c2, y2 y 5c1e 5t 5(y1 c2), parallel straight lines 5y1 y2 const 12. y1 y A cos 1_ 4t B sin 1_ 4t, y2 y 1_ 4A sin 1_ 4t 1_ 4B cos 1_ 4t; hence y1 2 16y2 2 (A2 B2) (cos2 1_4t sin 2 1_ 4t) const. Ellipses. 14. y1 dy1/d, y2 dy2/d, reversal of the direction of motion. To get the usual form, we have to multiply the transformed system by 1. This amounts to multiplying the matrix by 1, changing p into p (changing stability into instability and conversely when p 0) but leaving q and unchanged. 16. We have p 0 and q 0. We get p 2k 0 and q (a11 k)(a22 k) a12a21 q k(a11 a22) k 2 q k2 0 and p2 4q (2k)2 4(q k2) 4q 0. This gives a spiral point, which is stable and attractive if k 0 and unstable if k 0. Instructor's Manual 79 im04.qxd 9/21/05 11:08 AM Page 79 SECTION 4.5. Qualitative Methods for Nonlinear Systems, page 151 Purpose. As a most important step, in this section we extend phase plane methods from linear to nonlinear systems and nonlinear ODEs. Main Content Critical points of nonlinear systems Their discussion by linearization Transformation of single autonomous ODEs Applications of linearization and transformation techniques Important Concepts and Facts Linearized system (3), condition for applicability Linearization of pendulum equations Self-sustained oscillations, van der Pol equation Short Courses. Linearization at different critical points seems the main issue that the student is supposed to understand and handle practically. Examples 1 and 2 may help students to gain skill in that technique. The other material can be skipped without loss of continuity. Some Details on Content This section is very important, because from it the student should learn not only techniques (linearization, etc.) but also the fact that phase plane methods are particularly powerful and important in application to systems or single ODEs that cannot be solved explicitly. The student should also recognize that it is quite surprising how much information these methods can give. This is demonstrated by the pendulum equation (Examples 1 and 2) for a relatively simple system, and by the famous van der Pol equation for a single ODE, which has become a prototype for self-sustained oscillations of electrical systems of various kinds. We also discuss the famous Lotka–Volterra predator–prey model. For the Rayleigh and Duffing equations, see the problem set. SOLUTIONS TO PROBLEM SET 4.5, page 158 2. Writing the system in the form y1 y1(4 y1) y2 y2 we see that the critical points are (0, 0) and (4, 0). For (0, 0) the linearized system is y1 4y1 y2 y2. The matrix is [ ] . Hence p 5, q 4, 25 16 9. This shows that the critical point at (0, 0) is an unstable node. 0 1 4 0 80 Instructor's Manual im04.qxd 9/21/05 11:08 AM Page 80 8. y1 y2, y2 y1(9 y1). (0, 0) is a critical point. The linearized system at (0, 0) is with matrix [ ] for which p 0 and q 9 0, so that we have a center. A second critical point is at (9, 0). The transformation is y1 9 y1, y2 y2. This gives the transformed system y1 y2 y2 (9 y1)(y1). Its linearization is with matrix [ ] for which q 9 0, so that we have a saddle point. 10. The system is y1 y2 y2 sin y1. The critical points occur at (n, 0), n 0, 1, • • • , where the sine is zero and y2 0. At (0, 0) the linearized system is y1 y2 y2 y1. Its matrix is [ ] . Hence p 0 and q 1, so that we have a center. By periodicity, the points (2n, 0) are centers. We consider (, 0). The transformation is y1 y1, y2 y2. This gives the transformed system y1 y2 y2 sin (y1) sin y1. Linearization gives the system y1 y2 y2 y1. The matrix is [ ] . Hence q 1. This is a saddle point. By periodicity the critical points at (2n, 0) are saddle points. 1 0 0 1 1 0 0 1 1 0 0 9 y1 y2 y2 9y1 1 0 0 9 y1 y2 y2 9y1 Instructor's Manual 83 im04.qxd 9/21/05 11:08 AM Page 83 12. The system is y1 y2 y2 y1(2 y1) y2. The critical points are (0, 0) and (2, 0). At (0, 0) the linearized system is y1 y2 y2 2y1 y2. Its matrix is [ ] . Hence p 1, q 2, 1 8 0. This gives a stable and attractive spiral point. At (2, 0) the transformation is y1 2 y1, y2 y2. This gives the transformed system y1 y2 y2 (2 y1)(y1) y2. Its linearization is y1 y2 y2 2y1 y2. Its matrix is [ ] . Hence p 1, q 2 0. This gives a saddle point. 14. The system is (a) y1 y2 (b) y2 4y1 y1 3. Multiply the left side of (a) by the right side of (b) and the right side of (a) by the left side of (b), obtaining y2y2 (4y1 y1 3)y1. Integrate and multiply by 2: y2 2 4y1 2 1_2y1 4 c*. Setting c* c2/2 8, write this as y2 2 1_2(c 4 y1 2)(c 4 y1 2). Some of these curves are shown in the figure. 16. The system is y1 y2 y2 2y2 4y1 y1 3 y1(y1 2)(y1 2) 2y2. Hence the critical points are (0, 0), (2, 0), (2, 0). 1 1 0 2 1 1 0 2 84 Instructor's Manual im04.qxd 9/21/05 11:08 AM Page 84 At (0, 0) the linearized system is with matrix [ ] . Hence p 2, q 4 0, which gives a saddle point. At (2, 0) the transformation is y1 2 y1, y2 y2. This gives the system y1 y2 y2 (2 y1)y1(y1 4) 2y2 linearized with matrix [ ] . Hence p 2, q 8, 4 32 0. This gives a stable and attractive spiral point. At (2, 0) the transformation is y1 2 y1, y2 y2. This gives the system y1 y2 y2 (2 y1)(4 y1)y1 2y2. Linearization gives with matrix [ ] as before, so that we obtain another stable and attractive spiral point. Note the similarity to the situation in the case of the undamped and damped pendulum. 18. A limit cycle is approached by trajectories (from inside and outside). No such approach takes place for a closed trajectory. 20. Team Project. (a) Unstable node if 2, unstable spiral point if 2 0, center if 0, stable and attractive spiral point if 0 2, stable and attractive node if 2. 1 2 0 8 y1 y2 y2 8y1 2y2 1 2 0 8 y1 y2 y2 8y1 2y2 1 2 0 4 y1 y2 y2 4y1 2y2 Instructor's Manual 85 y1 2–2 c = 3 c = 4 c = 5 y2 Section 4.5. Problem 14 im04.qxd 9/21/05 11:08 AM Page 85 6. y(h) can be obtained from Example 5 in Sec. 4.3 in the form y(h) c1 [ ] e(1i)t c2 [ ] e(1i)t [ ] where A c1 c2 and B i(c1 c2). y(p) is obtained by the method of undetermined coefficients, starting from y(p) [a1 a2] Te2t. Differentiation and substitution into the given nonhomogeneous system gives, in components, 2a1e 2t (a1 a2)e 2t e2t 2a2e 2t (a1 a2)e 2t e2t. Dropping e2t, we have 2a1 a1 a2 1 2a2 a1 a2 1. The solution is a1 1, a2 0. We thus obtain the answer y1 e t(A cos t B sin t) e2t y2 e t(B cos t A sin t). It is remarkable that y2 is the same as for the homogeneous system. 8. The matrix of the homogeneous system [ ] has the eigenvalues 8 and 8 with eigenvectors [3 1]T and [1 3]T, respectively. Hence a general solution of the homogeneous system is y(h) c1 [ ] e8t c2 [ ] e8t. We determine a particular solution y(p) of the nonhomogeneous system by the method of undetermined coefficients. We start from y(p) [ ] t2 [ ] t [ ] . Differentiation and substitution gives, in terms of components, 2a1t b1 (10a1 6a2)t 2 (10b1 6b2)t 10k1 6k2 10t 2 10t 10 2a2t b2 (6a1 10a2)t 2 (6b1 10b2)t 6k1 10k2 6t 2 20t 4. By equating the sum of the coefficients of t2 in each equation to zero we get 10a1 6a2 10 0 6a1 10a2 6 0. k1 k2 b1 b2 a1 a2 1 3 3 1 6 10 10 6 et(A cos t B sin t) et(B cos t A sin t) 1 i 1 i 88 Instructor's Manual im04.qxd 9/21/05 11:08 AM Page 88 The solution is a1 1, a2 0. Similarly for the terms in t we obtain 2a1 10b1 6b2 10 2a2 6b1 10b2 20. The solution is b1 0, b2 2. Finally, for the constant terms we obtain b1 10k1 6k2 10 b2 6k1 10k2 4. The solution is k1 1, k2 0. The answer is y y (h) y(p) with y(h) given above and y(p) [ ] . 12. The matrix of the homogeneous system [ ] has the eigenvalues 2i and 2i and eigenvectors [2 i]T and [2 i]T, respectively. Hence a complex general solution is y(h) c1 [ ] e2it c2 [ ] e2it. By Euler's formula this becomes, in components, y1 (h) (2c1 2c2) cos 2t i(2c1 2c2) sin 2t y2 (h) (ic1 ic2) cos 2t (c1 c2) sin 2t. Setting A c1 c2 and B i(c1 c2), we can write y1 (h) 2A cos 2t 2B sin 2t y2 (h) B cos 2t A sin 2t. Before we can consider the initial conditions, we must determine a particular solution y(p) of the given system. We do this by the method of undetermined coefficients, setting y1 (p) a1e t b1e t y2 (p) a2e t b2e t. Differentiation and substitution gives a1e t b1e t 4a2e t 4b2e t 5et a2e t b2e t a1e t b1e t 20et. Equating the coefficients of et on both sides, we get a1 4a2 5, a2 a1, hence a1 1, a2 1. Equating the coefficients of et, we similarly obtain b1 4b2, b2 b1 20, hence b1 16, b2 4. 2 i 2 i 4 0 0 1 t2 1 2t Instructor's Manual 89 im04.qxd 9/21/05 11:08 AM Page 89 Hence a general solution of the given nonhomogeneous system is y1 2A cos 2t 2B sin 2t e t 16et y2 B cos 2t A sin 2t e t 4et. From this and the initial conditions we obtain y1(0) 2A 1 16 1, y2(0) B 1 4 0. The solution is A 8, B 3. This gives the answer (the solution of the initial value problem) y1 16 cos 2t 6 sin 2t e t 16et y2 3 cos 2t 8 sin 2t e t 4et. 14. The matrix of the homogeneous system [ ] has the eigenvalues 1 and 2, with eigenvectors [1 1]T and [4 1]T, respectively. Hence a general solution of the homogeneous system is y(h) c1 [ ] et c2 [ ] e2t. We determine a particular solution y(p) of the nonhomogeneous system by the method of undetermined coefficients, starting from y(p) [ ] cos t [ ] sin t. Differentiation and substitution gives, in terms of components, (1) A1 sin t B1 cos t 3A1 cos t 3B1 sin t 4A2 cos t 4B2 sin t 20 cos t (2)A2 sin t B2 cos t A1 cos t B1 sin t 2A2 cos t 2B2 sin t. The coefficients of the cosine terms in (1) give B1 3A1 4A2 20 and for the sine terms we get A1 3B1 4B2. The coefficients of the cosine terms in (2) give B2 A1 2A2 and for the sine terms we get A2 B1 2B2. Hence A1 14, A2 6, B1 2, B2 2. Consequently, a general solution of the given system is, in terms of components, y1 c1e t 4c2e 2t 14 cos t 2 sin t y2 c1e t c2e 2t 6 cos t 2 sin t. B1 B2 A1 A2 4 1 1 1 4 2 3 1 90 Instructor's Manual im04.qxd 9/21/05 11:08 AM Page 90